Quadrature Amplitude Modulation (QAM) Transmitter

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Lecture 15

Introduction

- **Digital Pulse Amplitude Modulation (PAM) modulates digital information onto amplitude of pulse and may be later modulated by sinusoid**
- **Digital Quadrature Amplitude Modulation (QAM) is two-dimensional extension of digital PAM that requires sinusoidal modulation**
- **Digital QAM modulates digital information onto pulses that are modulated onto**
  - Amplitudes of a sine and a cosine, or equivalently
  - Amplitude and phase of single sinusoid

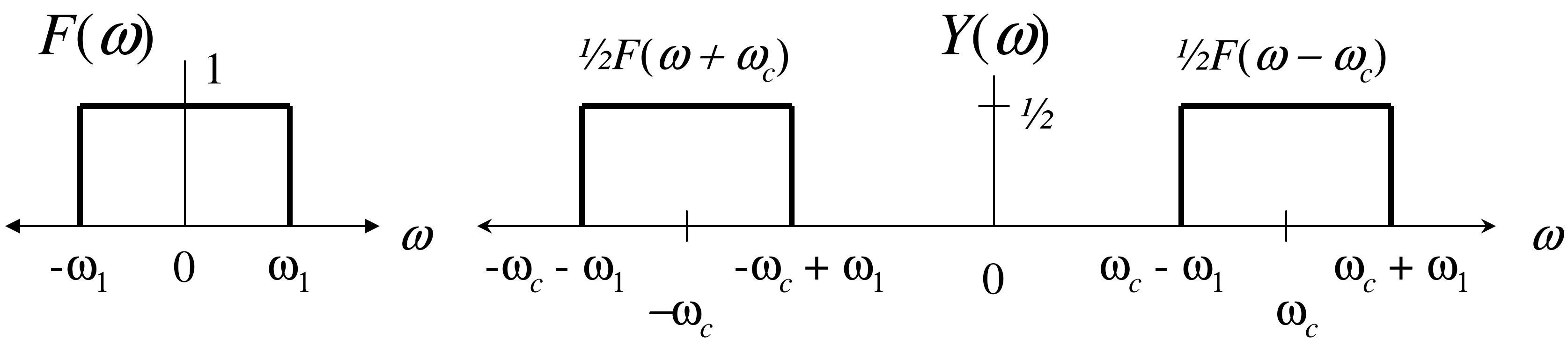
Amplitude Modulation by Cosine

- **Example:** $y(t) = f(t)\cos(\omega_c t)$

  Assume $f(t)$ is an ideal lowpass signal with bandwidth ω_1

  Assume $\omega_1 < \omega_c$

  $Y(\omega)$ is real-valued if $F(\omega)$ is real-valued



- **Demodulation: modulation then lowpass filtering**
- **Similar derivation for modulation with** $\sin(\omega_0 t)$

Review

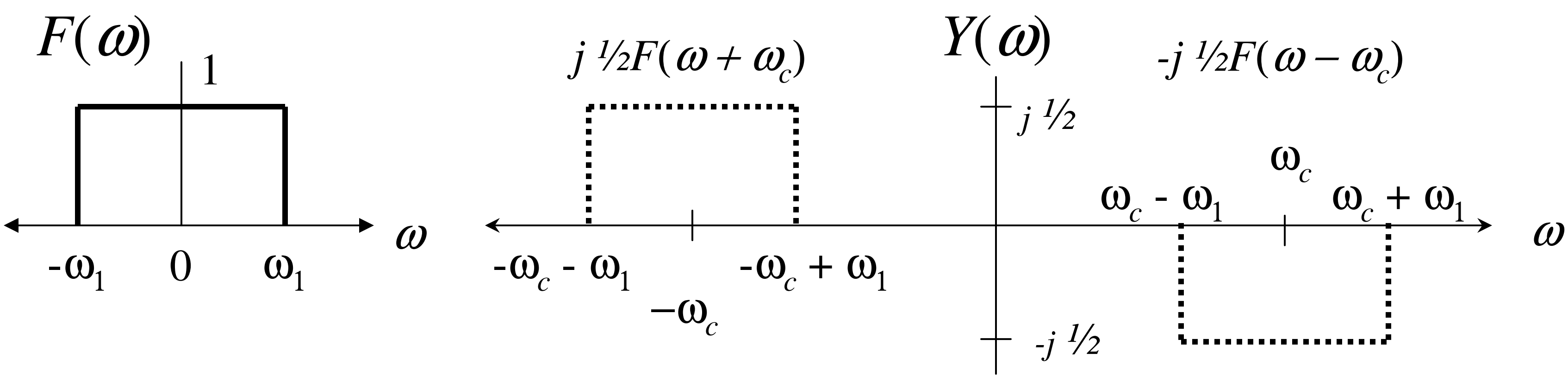
Amplitude Modulation by Sine

- **Example:** $y(t) = f(t)\sin(\omega_c t)$

  Assume $f(t)$ is an ideal lowpass signal with bandwidth ω_1

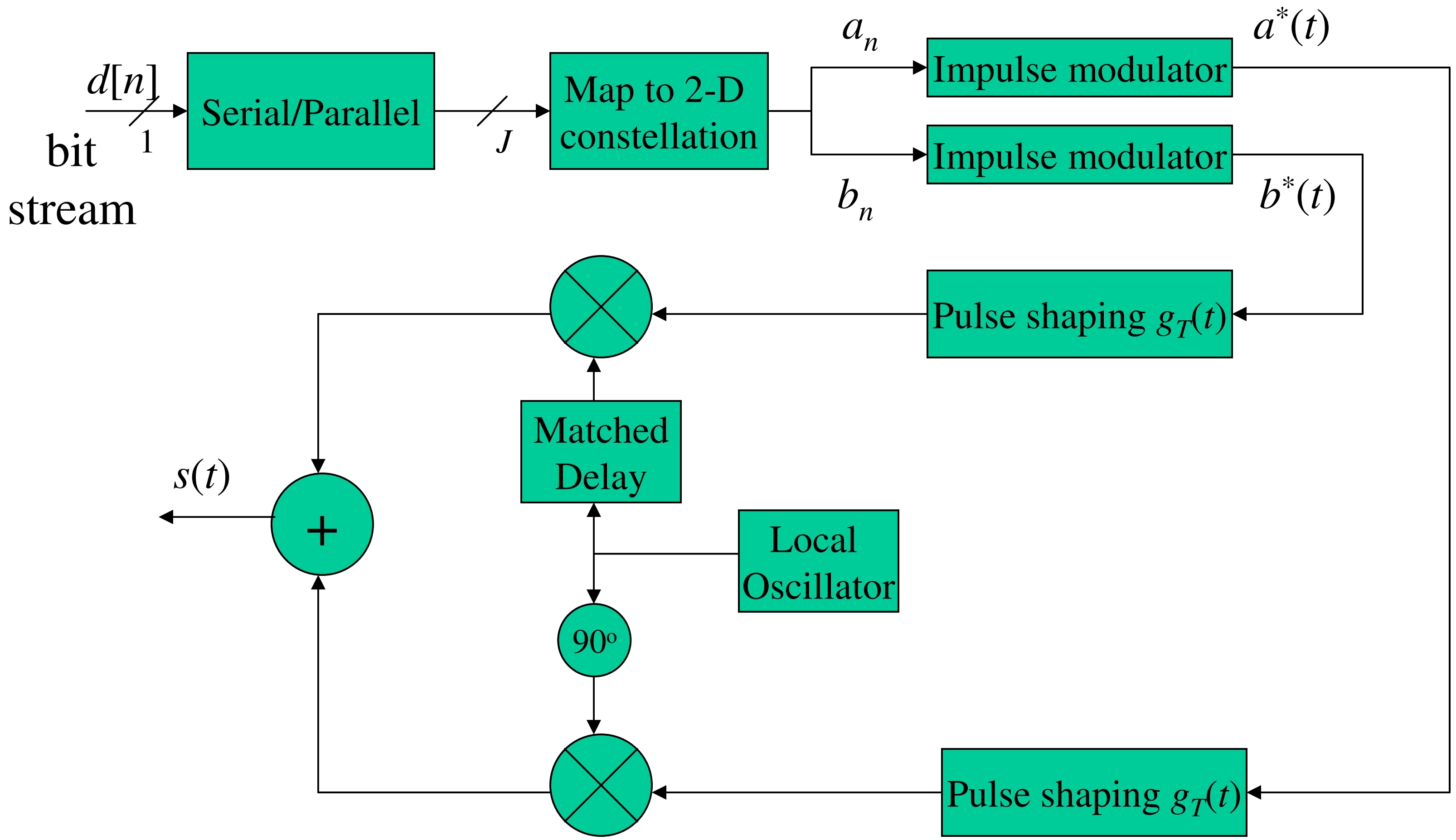
  Assume $\omega_1 < \omega_c$

  $Y(\omega)$ is imaginary-valued if $F(\omega)$ is real-valued



- **Demodulation: modulation then lowpass filtering**

Digital QAM Modulator



Matched delay matches delay through 90º phase shifter

Phase Shift by 90 Degrees

- **90^o phase shift performed by Hilbert transformer**

cosine => sine $$\cos(2\pi f_0 t) \Rightarrow \frac{1}{2}\delta(f + f_0) + \frac{1}{2}\delta(f - f_0)$$

sine => – cosine $$\sin(2\pi f_0 t) \Rightarrow \frac{j}{2}\delta(f + f_0) - \frac{j}{2}\delta(f - f_0)$$

- **Frequency response of ideal Hilbert transformer:**

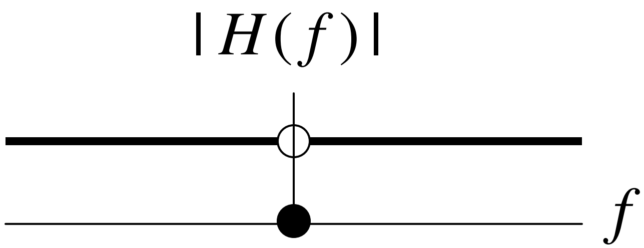
$$\boxed{H(f) = -j\,\mathrm{sgn}(f)}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Hilbert Transformer

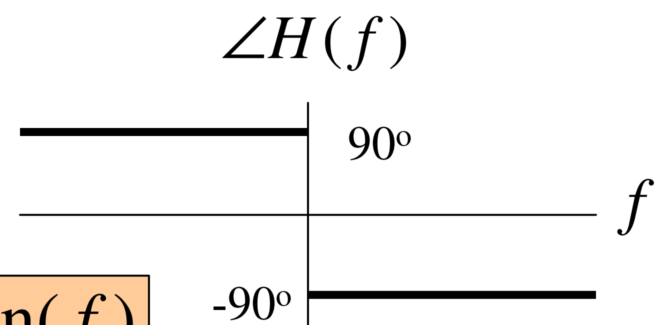
- **Magnitude response**

All pass except at origin



- **Phase response**

Piecewise constant



$$\boxed{H(f) = -j\,\mathrm{sgn}(f)}$$

- **For $f_c > 0$**

$$\cos(2\pi f_c t + \frac{\pi}{2}) = \sin(2\pi f_c t)$$

- **For $f_c < 0$**

$$\cos(2\pi f_c t - \frac{\pi}{2}) = \cos(-(2\pi f_c t + \frac{\pi}{2}))$$

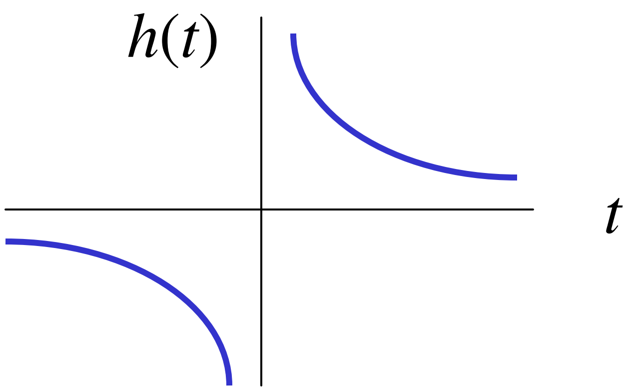
$$= \cos(2\pi(-f_c)t + \frac{\pi}{2}) = \sin(2\pi(-f_c)t)$$

Hilbert Transformer

- **Continuous-time ideal Hilbert transformer**

$$H(f) = -j\,\mathrm{sgn}(f)$$

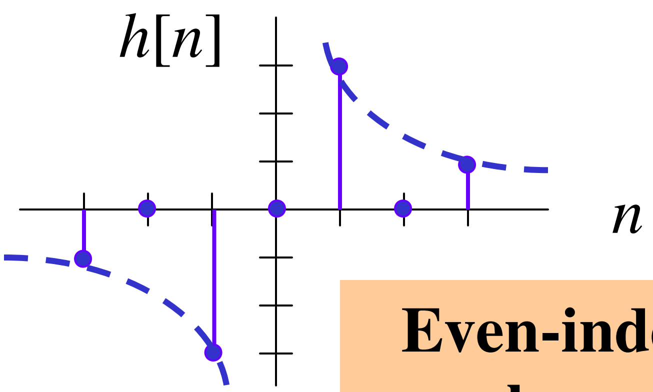
$$h(t) = \begin{cases} 1/(\pi\, t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$



- **Discrete-time ideal Hilbert transformer**

$$H(\omega) = -j\,\mathrm{sgn}(\omega)$$

$$h[n] = \begin{cases} \dfrac{2}{\pi}\dfrac{\sin^2(\pi n/2)}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$



Even-indexed samples are zero

Discrete-Time Hilbert Transformer

- **Approximate by odd-length linear phase FIR filter**

  Truncate response to $2\,L + 1$ samples: L samples left of origin, L samples right of origin, and origin

  Shift truncated impulse response by L samples to right to make it causal

  L is odd because every other sample of impulse response is 0

- **Linear phase FIR filter of length N has same phase response as a delay of length $(N$-1)/2**

  $(N\text{-}1)/2$ is an integer when N is odd (here $N = 2\,L + 1$)

- **How would you make sure that delay from local oscillator to sine modulator is equal to delay from local oscillator to cosine modulator?**

Performance Analysis of PAM

- **If we sample matched filter output at correct time instances, nT_{sym}, without any ISI, received signal**

$$x(nT_{sym}) = s(nT_{sym}) + v(nT_{sym})$$

$$\boldsymbol{v(nT) \sim N(0;\, \sigma^2/T_{sym})}$$

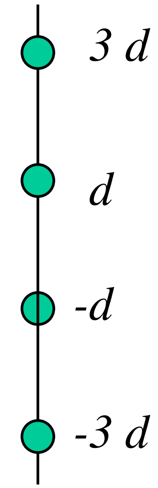
where the signal component is

$$s(nT_{sym}) = a_n = (2i - 1)d \quad \text{for } i = -M/2+1, \ldots, M/2$$

$v(t)$ output of matched filter $G_r(\omega)$ for input of channel additive white Gaussian noise $N(0;\, \sigma^2)$

$G_r(\omega)$ passes frequencies from $-\omega_{sym}/2$ to $\omega_{sym}/2$, where $\omega_{sym} = 2\,\pi\, f_{sym} = 2\pi \,/\, T_{sym}$

- **Matched filter has impulse response $g_r(t)$**



4-PAM

Performance Analysis of PAM

$$v(nT) = \int_{-\infty}^{\infty} g_r(\tau) w(nT-\tau) d\tau$$

Filtered noise

$\boldsymbol{T = T_{sym}}$

$$E\{v^2(nT)\} = E\left\{\left[\int_{-\infty}^{\infty} g_r(\tau) w(nT-\tau) d\tau\right]^2\right\}$$

Noise power

$$= E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_T(\tau_1) w(nT-\tau_1) g_T(\tau_2) w(nT-\tau_2) d\tau_1 d\tau_2\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_T(\tau_1) g_T(\tau_2) \underbrace{E\{w(nT-\tau_1) w(nT-\tau_2)\}}_{\sigma^2 \delta(\tau_1-\tau_2)} d\tau_1 d\tau_2$$

$$= \sigma^2 \int_{-\infty}^{\infty} g_r^2(\tau) d\tau = \sigma^2 \frac{1}{2\pi} \int_{-\omega_{sym}/2}^{\omega_{sym}/2} G_r^2(\omega) d\omega = \frac{\sigma^2}{T}$$

Performance Analysis of PAM

- **Decision error for inner points**

$$P_I(e) = P(|v(nT)| > d) = 2\,Q\left(\frac{d}{\sigma}\sqrt{T}\right)$$

- **Decision error for outer points**

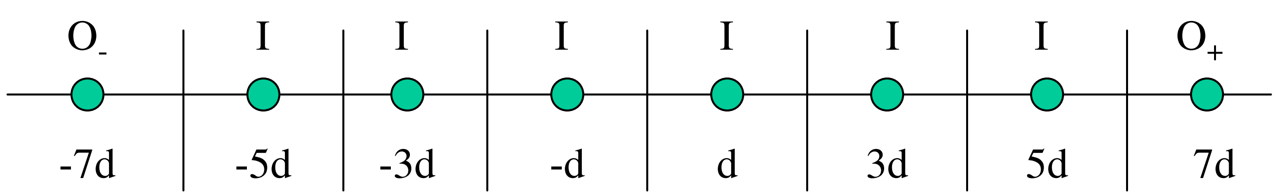
$$P_{O_-}(e) = P(v(nT) > d) = Q\left(\frac{d}{\sigma}\sqrt{T}\right)$$

$$P_{O_+}(e) = P(v(nT) < -d) = P(v(nT) > d)Q\left(\frac{d}{\sigma}\sqrt{T}\right)$$

- **Symbol error probability**

$$P(e) = \frac{M-2}{M} P_I(e) + \frac{1}{M} P_{O_+}(e) + \frac{1}{M} P_{O_-}(e) = \frac{2(M-1)}{M} Q\left(\frac{d}{\sigma}\sqrt{T}\right)$$

8-PAM Constellation



Performance Analysis of QAM

- **Received QAM signal**

$$x(nT) = s(nT) + v(nT)$$

- **Information signal $s(nT)$**

$$s(nT) = a_n + j\, b_n = (2i-1)d + j\,(2k-1)d$$

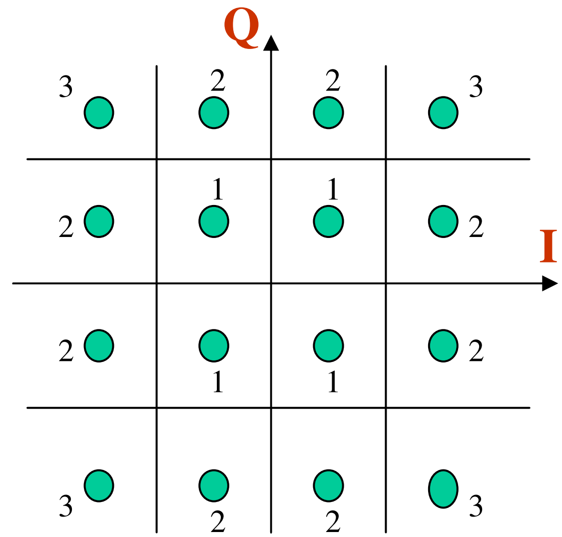
where $i,k \in$ { -1, 0, 1, 2 } for 16-QAM

- **Noise, $v_I(nT)$ and $v_Q(nT)$ are independent Gaussian random variables ~ $N(0; \sigma^2/T)$**

$$v(nT) = v_I(nT) + j\, v_Q(nT)$$

Performance Analysis of QAM

- **Type 1 correct detection**



16-QAM

$$P_1(c) = P(|v_I(nT)| < d \;\&\; |v_Q(nT)| < d)$$

$$= P(|v_I(nT)| < d)P(|v_Q(nT)| < d)$$

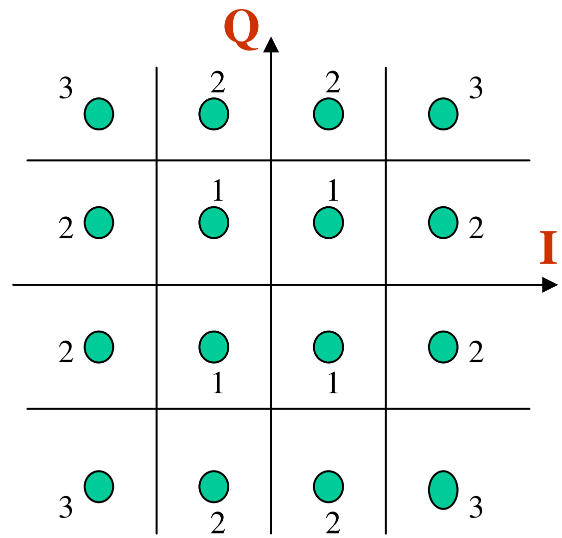
$$= (1 - \underbrace{P(|v_I(nT)| > d)}_{2Q(\frac{d}{\sigma}\sqrt{T})})(1 - \underbrace{P(|v_Q(nT)| > d)}_{2Q(\frac{d}{\sigma}\sqrt{T})})$$

$$= (1 - 2Q(\frac{d}{\sigma}\sqrt{T}))^2$$

Performance Analysis of QAM

- **Type 2 correct detection**

$$P_2(c) = P(v_I(nT) < d \;\&\; |v_Q(nT)| < d)$$
$$= P(v_I(nT) < d)P(|v_Q(nT)| < d)$$
$$= (1 - 2Q(\frac{d}{\sigma}\sqrt{T}))(1 - Q(\frac{d}{\sigma}\sqrt{T}))$$



16-QAM

- **Type 3 correct detection**

$$P_3(c) = P(v_I(nT) < d \;\&\; v_Q(nT) > -d)$$
$$= P(v_I(nT) < d)P(v_Q(nT) > -d)$$
$$= (1 - Q(\frac{d}{\sigma}\sqrt{T}))^2$$

Performance Analysis of QAM

- **Probability of correct detection**

$$P(c) = \frac{4}{16}(1-2Q(\frac{d}{\sigma}\sqrt{T}))^2 + \frac{4}{16}(1-Q(\frac{d}{\sigma}\sqrt{T}))^2$$

$$+\frac{8}{16}(1-2Q(\frac{d}{\sigma}\sqrt{T}))(1-Q(\frac{d}{\sigma}\sqrt{T}))$$

$$= 1-3Q(\frac{d}{\sigma}\sqrt{T}) + \frac{9}{4}Q^2(\frac{d}{\sigma}\sqrt{T})$$

- **Symbol error probability**

$$P(e) = 1-P(c) = 3Q(\frac{d}{\sigma}\sqrt{T}) - \frac{9}{4}Q^2(\frac{d}{\sigma}\sqrt{T})$$

Average Power Analysis

- **PAM and QAM signals are deterministic**
- **For a deterministic signal $p(t)$, instantaneous power is $|p(t)|^2$**
- **4-PAM constellation points: { $-3\,d$, $-d$, d, $3\,d$ }**
  - Total power $9\,d^2 + d^2 + d^2 + 9\,d^2 = 20\,d^2$
  - Average power per symbol $5\,d^2$
- **4-QAM constellation points: { $d + j\,d$, $-d + j\,d$, $d - j\,d$, $-d - j\,d$ }**
  - Total power $2\,d^2 + 2\,d^2 + 2\,d^2 + 2\,d^2 = 8\,d^2$
  - Average power per symbol $2\,d^2$