Quadrature Amplitude Modulation (QAM) Transmitter

Prof. Brian L. Evans
Dept. of Electrical and Computer Engineering
The University of Texas at Austin

Lecture 15
Introduction

- Digital Pulse Amplitude Modulation (PAM) modulates digital information onto amplitude of pulse and may be later modulated by sinusoid.
- Digital Quadrature Amplitude Modulation (QAM) is two-dimensional extension of digital PAM that requires sinusoidal modulation.
- Digital QAM modulates digital information onto pulses that are modulated onto Amplitudes of a sine and a cosine, or equivalently Amplitude and phase of single sinusoid.
Amplitude Modulation by Cosine

- **Example:** \( y(t) = f(t) \cos(\omega_c t) \)

  Assume \( f(t) \) is an ideal lowpass signal with bandwidth \( \omega_1 \)

  Assume \( \omega_1 < \omega_c \)

  \( Y(\omega) \) is real-valued if \( F(\omega) \) is real-valued

- Demodulation: modulation then lowpass filtering

- Similar derivation for modulation with \( \sin(\omega_0 t) \)
Amplitude Modulation by Sine

- **Example:** \( y(t) = f(t) \sin(\omega_c t) \)
  
  Assume \( f(t) \) is an ideal lowpass signal with bandwidth \( \omega_1 \)
  
  Assume \( \omega_1 < \omega_c \)
  
  \( Y(\omega) \) is imaginary-valued if \( F(\omega) \) is real-valued

- **Demodulation:** modulation then lowpass filtering
Digital QAM Modulator

Matched delay matches delay through 90° phase shifter
Phase Shift by 90 Degrees

• $90^\circ$ phase shift performed by Hilbert transformer

\[
\begin{align*}
\cos(2\pi f_0 t) &\implies \frac{1}{2} \delta(f + f_0) + \frac{1}{2} \delta(f - f_0) \\
\sin(2\pi f_0 t) &\implies \frac{j}{2} \delta(f + f_0) - \frac{j}{2} \delta(f - f_0)
\end{align*}
\]

• Frequency response of ideal Hilbert transformer:

\[H(f) = -j \text{sgn}(f)\]

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]
Hilbert Transformer

- **Magnitude response**
  All pass except at origin
  \[ |H(f)| \]

- **Phase response**
  Piecewise constant
  \[ \angle H(f) \]

- **For** \( f_c > 0 \)
  \[
  \cos(2\pi f_c t + \frac{\pi}{2}) = \sin(2\pi f_c t)
  \]

- **For** \( f_c < 0 \)
  \[
  \cos(2\pi f_c t - \frac{\pi}{2}) = \cos(-2\pi f_c t + \frac{\pi}{2})
  = \cos(2\pi(-f_c) t + \frac{\pi}{2}) = \sin(2\pi(-f_c) t)
  \]
  
  \[ H(f) = -j \text{sgn}(f) \]
Hilbert Transformer

- Continuous-time ideal Hilbert transformer

\[ H(f) = -j \text{sgn}(f) \]

\[ h(t) = \begin{cases} 
\frac{1}{\pi t} & \text{if } t \neq 0 \\
0 & \text{if } t = 0 
\end{cases} \]

- Discrete-time ideal Hilbert transformer

\[ H(\omega) = -j \text{sgn}(\omega) \]

\[ h[n] = \begin{cases} 
\frac{2 \sin^2(\pi n/2)}{\pi n} & \text{if } n \neq 0 \\
0 & \text{if } n = 0 
\end{cases} \]

Even-indexed samples are zero
Discrete-Time Hilbert Transformer

• Approximate by odd-length linear phase FIR filter
  Truncate response to $2L + 1$ samples: $L$ samples left of origin, $L$ samples right of origin, and origin
  Shift truncated impulse response by $L$ samples to right to make it causal
  $L$ is odd because every other sample of impulse response is 0

• Linear phase FIR filter of length $N$ has same phase response as a delay of length $(N-1)/2$
  $(N-1)/2$ is an integer when $N$ is odd (here $N = 2L + 1$)

• How would you make sure that delay from local oscillator to sine modulator is equal to delay from local oscillator to cosine modulator?
Performance Analysis of PAM

- If we sample matched filter output at correct time instances, \( nT_{sym} \), without any ISI, received signal

\[
x(nT_{sym}) = s(nT_{sym}) + v(nT_{sym})
\]

where the signal component is

\[
s(nT_{sym}) = a_n = (2i - 1)d \quad \text{for } i = -M/2+1, \ldots, M/2
\]

\( v(t) \) output of matched filter \( G_r(\omega) \) for input of

channel additive white Gaussian noise \( N(0; \sigma^2) \)

\( G_r(\omega) \) passes frequencies from \(-\omega_{sym}/2\) to \(\omega_{sym}/2\),

where \( \omega_{sym} = 2 \pi f_{sym} = 2\pi / T_{sym} \)

- Matched filter has impulse response \( g_r(t) \)
Performance Analysis of PAM

\[ v(nT) = \int_{-\infty}^{\infty} g_r(\tau)w(nT - \tau)d\tau \]

**Filtered noise**

\[ T = T_{\text{sym}} \]

\[ E\{v^2(nT)\} = E\left\{ \left[ \int_{-\infty}^{\infty} g_r(\tau)w(nT - \tau)d\tau \right]^2 \right\} \]

**Noise power**

\[ = E\left\{ \int_{-\infty}^{\infty} g_T(\tau_1)w(nT - \tau_1)g_T(\tau_2)w(nT - \tau_2)d\tau_1d\tau_2 \right\} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_T(\tau_1)g_T(\tau_2)E\{w(nT - \tau_1)w(nT - \tau_2)\}d\tau_1d\tau_2 \]

\[ = \sigma^2 \int_{-\infty}^{\infty} g_r^2(\tau)d\tau = \sigma^2 \frac{1}{2\pi} \int_{-\omega_{\text{sym}}/2}^{\omega_{\text{sym}}/2} G_r^2(\omega)d\omega = \frac{\sigma^2}{T} \]
Performance Analysis of PAM

- Decision error for inner points
  \[ P_i(e) = P(|v(nT)| > d) = 2Q\left(\frac{d}{\sigma}\sqrt{T}\right) \]

- Decision error for outer points
  \[ P_{o-}(e) = P(v(nT) > d) = Q\left(\frac{d}{\sigma}\sqrt{T}\right) \]
  \[ P_{o+}(e) = P(v(nT) < -d) = P(v(nT) > d)Q\left(\frac{d}{\sigma}\sqrt{T}\right) \]

- Symbol error probability
  \[ P(e) = \frac{M-2}{M} P_i(e) + \frac{1}{M} P_{o+}(e) + \frac{1}{M} P_{o-}(e) = \frac{2(M-1)}{M} Q\left(\frac{d}{\sigma}\sqrt{T}\right) \]

8-PAM Constellation

- \( O_- \) at -7d
- \( I \) at -5d
- \( I \) at -3d
- \( I \) at -d
- \( I \) at d
- \( I \) at 3d
- \( I \) at 5d
- \( O_+ \) at 7d
Performance Analysis of QAM

• Received QAM signal
  \[ x(nT) = s(nT) + v(nT) \]

• Information signal \( s(nT) \)
  \[ s(nT) = a_n + j b_n = (2i - 1)d + j (2k - 1)d \]
  where \( i,k \in \{ -1, 0, 1, 2 \} \) for 16-QAM

• Noise, \( v_I(nT) \) and \( v_Q(nT) \) are independent Gaussian random variables \( \sim N(0; \sigma^2/T) \)
  \[ v(nT) = v_I(nT) + j v_Q(nT) \]
Performance Analysis of QAM

- Type 1 correct detection

\[
P_1(c) = P(\left| v_I(nT) \right| < d \& \left| v_Q(nT) \right| < d)
\]

\[
= P(\left| v_I(nT) \right| < d)P(\left| v_Q(nT) \right| < d)
\]

\[
= (1 - P(\left| v_I(nT) \right| > d))(1 - P(\left| v_Q(nT) \right| > d))
\]

\[
= (1 - 2Q\left( \frac{d}{\sigma}\sqrt{T} \right))^2
\]

16-QAM
Performance Analysis of QAM

- **Type 2 correct detection**

\[ P_2(c) = P(v_I(nT) < d \& |v_Q(nT)| < d) \]
\[ = P(v_I(nT) < d)P(|v_Q(nT)| < d) \]
\[ = (1 - 2Q\left(\frac{d}{\sigma}\sqrt{T}\right))(1 - Q\left(\frac{d}{\sigma}\sqrt{T}\right)) \]

- **Type 3 correct detection**

\[ P_3(c) = P(v_I(nT) < d \& v_Q(nT) > -d) \]
\[ = P(v_I(nT) < d)P(v_Q(nT) > -d) \]
\[ = (1 - Q\left(\frac{d}{\sigma}\sqrt{T}\right))^2 \]
Performance Analysis of QAM

- **Probability of correct detection**

\[
P(c) = \frac{4}{16} (1 - 2Q\left(\frac{d}{\sigma} \sqrt{T}\right))^2 + \frac{4}{16} (1 - Q\left(\frac{d}{\sigma} \sqrt{T}\right))^2
\]

\[
+ \frac{8}{16} (1 - 2Q\left(\frac{d}{\sigma} \sqrt{T}\right))(1 - Q\left(\frac{d}{\sigma} \sqrt{T}\right))
\]

\[
= 1 - 3Q\left(\frac{d}{\sigma} \sqrt{T}\right) + \frac{9}{4} Q^2\left(\frac{d}{\sigma} \sqrt{T}\right)
\]

- **Symbol error probability**

\[
P(e) = 1 - P(c) = 3Q\left(\frac{d}{\sigma} \sqrt{T}\right) - \frac{9}{4} Q^2\left(\frac{d}{\sigma} \sqrt{T}\right)
\]
Average Power Analysis

• PAM and QAM signals are deterministic
• For a deterministic signal \( p(t) \), instantaneous power is \( |p(t)|^2 \)
• 4-PAM constellation points: \{ -3d, -d, d, 3d \}
  – Total power \( 9d^2 + d^2 + d^2 + 9d^2 = 20d^2 \)
  – Average power per symbol \( 5d^2 \)
• 4-QAM constellation points: \{ d + jd, -d + jd, 
  d – jd, -d – jd \}
  – Total power \( 2d^2 + 2d^2 + 2d^2 + 2d^2 = 8d^2 \)
  – Average power per symbol \( 2d^2 \)