Digital Pulse Amplitude Modulation

- **Prof. Brian L. Evans**
- **Dept. of Electrical and Computer Engineering**
- **The University of Texas at Austin**

Lecture 14

Introduction

- Modulate $M = 2^J$ discrete messages or J bits of information into amplitude of signal
- If amplitude mapping changes at symbol rate of f_{sym}, then bit rate is $J\, f_{sym}$

$f_{sym} = 1 / T_{sym}$

- Conventional mapping of discrete messages to M uniformly space amplitudes

$$a_i = d\,(2i - 1) \qquad i = -\frac{M}{2} + 1, \ldots, 0, \ldots, \frac{M}{2}$$

- Pulse amplitude modulated (PAM) signal

$$s^*(t) = \sum_{k=-\infty}^{\infty} a_k\ \delta(t - k\, T_{sym})$$

No pulses overlap in time: requires infinite bandwidth

Pulse Shaping

- **Infinite bandwidth cannot be sent in practice**
- **Limit bandwidth by pulse shaping filter with impulse response $g_T(t)$**

$$s^*(t) = \sum_{k=-\infty}^{\infty} a_k \; g_{T_{sym}}\left(t - k\,T_{sym}\right)$$

At each time *t*, *k* is indexed over number of overlapping pulses

- ***L* samples per symbol duration**

n is symbol index

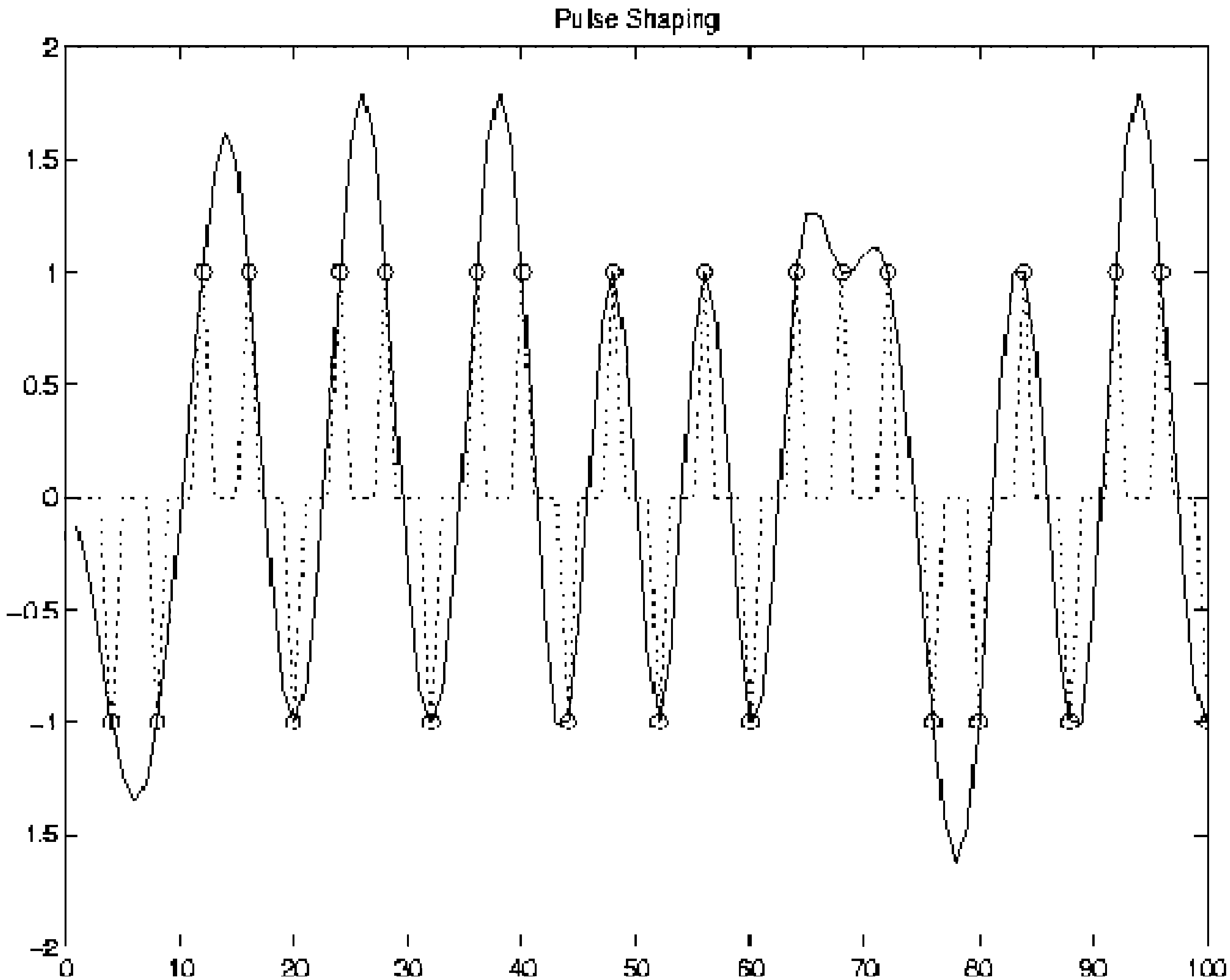
$$s^*\left(nT_{sym} + \frac{m}{L}T_{sym}\right) = \sum_{k=-\infty}^{\infty} a_k \; g_{T_{sym}}\left(nT_{sym} + \frac{m}{L}T_{sym} - kT_{sym}\right)$$

At indices *n* & *m*, *k* is indexed over number of overlapping pulses

m is sample index in a symbol: $m = 0, 1, 2, \ldots, L-1$.

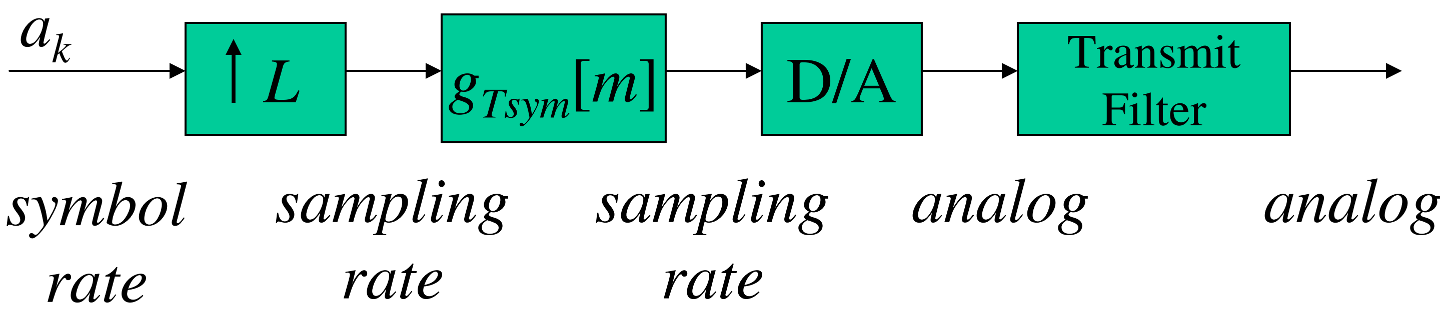
$$s^*(Ln+m) = \sum_{k=-\infty}^{\infty} a_k \; g_{T_{sym}}(Ln+m-Lk) = \sum_{k=-\infty}^{\infty} a_k \; g_{T_{sym}}(L(n-k)+m)$$

Pulse Shaping Example



2-PAM with Raised Cosine Pulse Shaping

Pulse Shaping Block Diagram



- **Upsampling by L denoted as $\uparrow L$**

  Outpus input sample followed by L-1 zeros

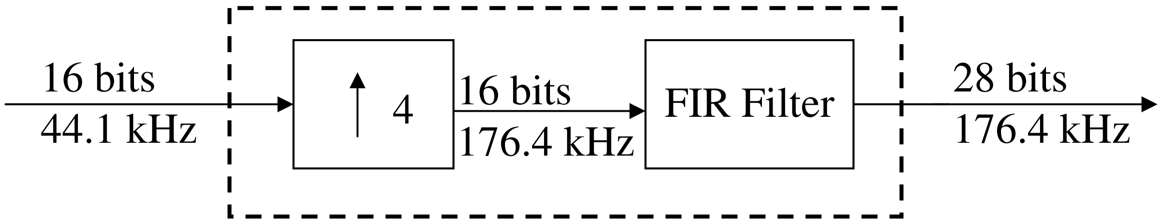
  Upsampling by converts symbol rate to sampling rate

- **Pulse shaping (FIR) filter $g_{Tsym}[m]$**

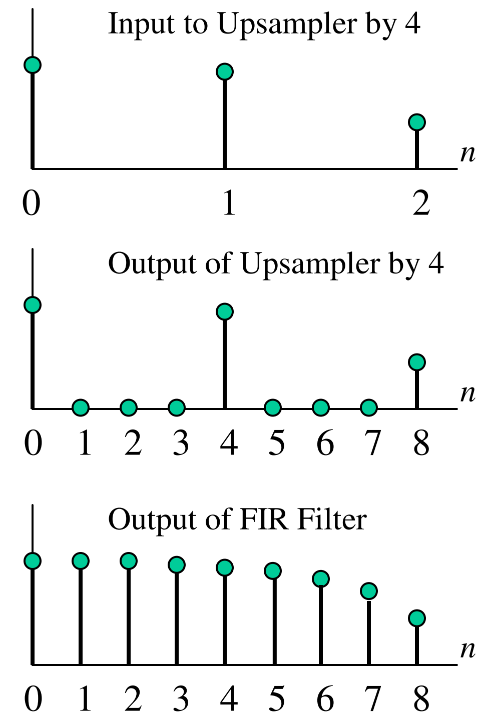
  Fills in zero values generated by upsampler

  Multiplies by zero most of time (L-1 out of every L times)

Digital Interpolation Example

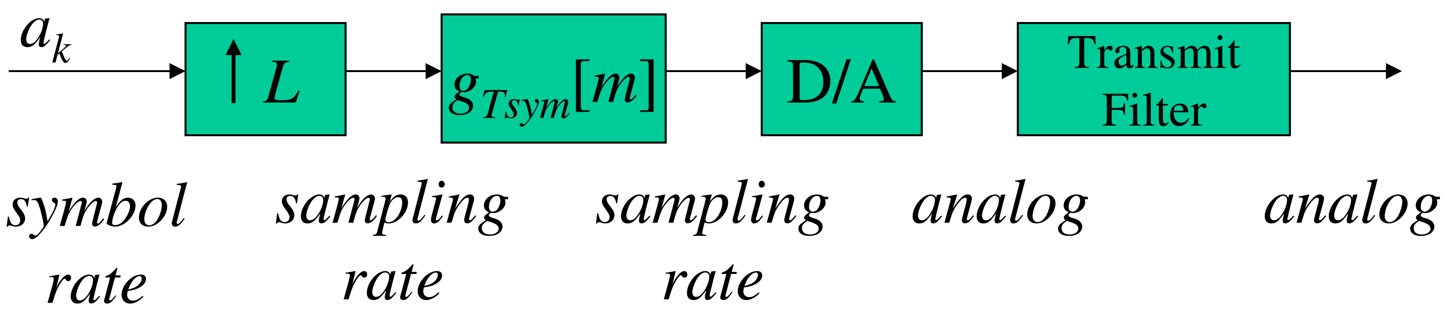


Digital 4x Oversampling Filter



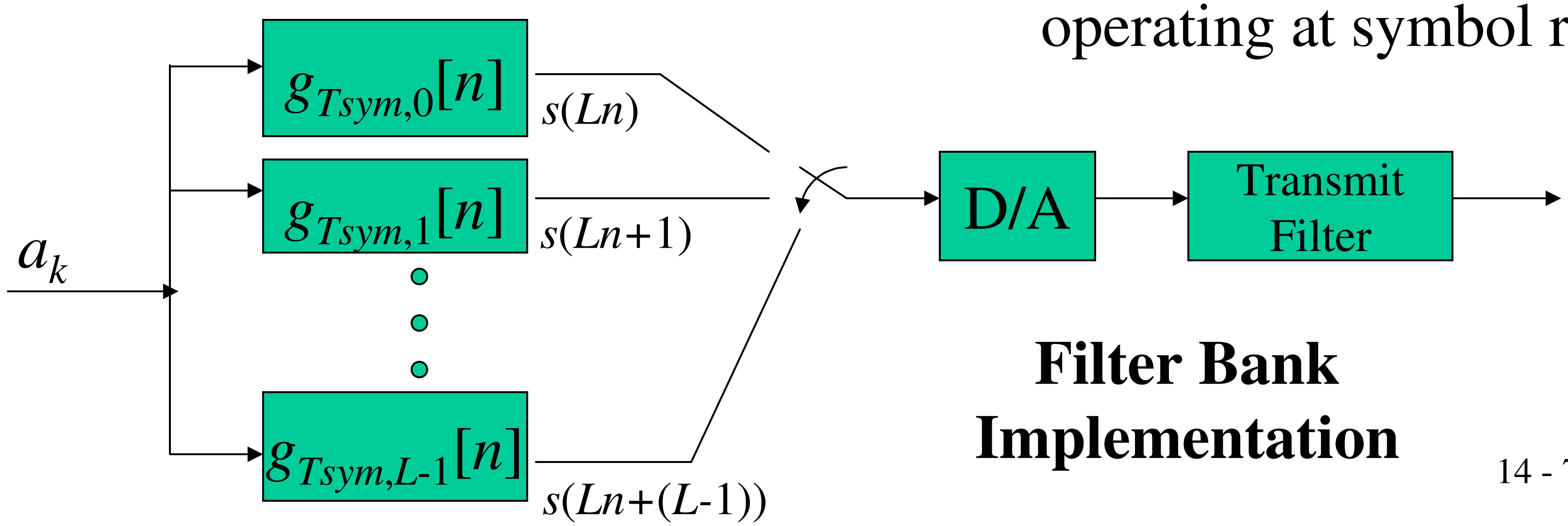
- **Upsampling by 4 (denoted by ↑4)**
  - Output input sample followed by 3 zeros
  - Four times the samples on output as input
  - Increases sampling rate by factor of 4
- **FIR filter performs interpolation**
  - Lowpass filter with stopband frequency $\omega_{stopband} \leq \pi / 4$
  - For $f_{\text{sampling}} = 176.4$ kHz, $\omega = \pi / 4$ corresponds to 22.05 kHz

Pulse Shaping Filter Bank



- **Simplify by avoiding multiplication by zero**

  Split the long pulse shaping filter into L short polyphase filters operating at symbol rate



Filter Bank Implementation

Pulse Shaping Filter Bank Example

- **Pulse length 24 samples and L = 4 samples/symbol**

$$s^*(Ln+m) = \sum_{k=n-2}^{n+3} a_k \; g_T(L(n-k)+m)$$

Six pulses contribute to each output sample

- **Derivation in Tretter's manual,**

$$s^*\left(nT_{sym} + \frac{m}{L}T_{sym}\right) = \sum_{k=n-2}^{n+3} a_k \; g_T\left(nT_{sym} + \frac{m}{L}T_{sym} - kT_{sym}\right) \qquad m = 0, 1, \ldots, L-1$$

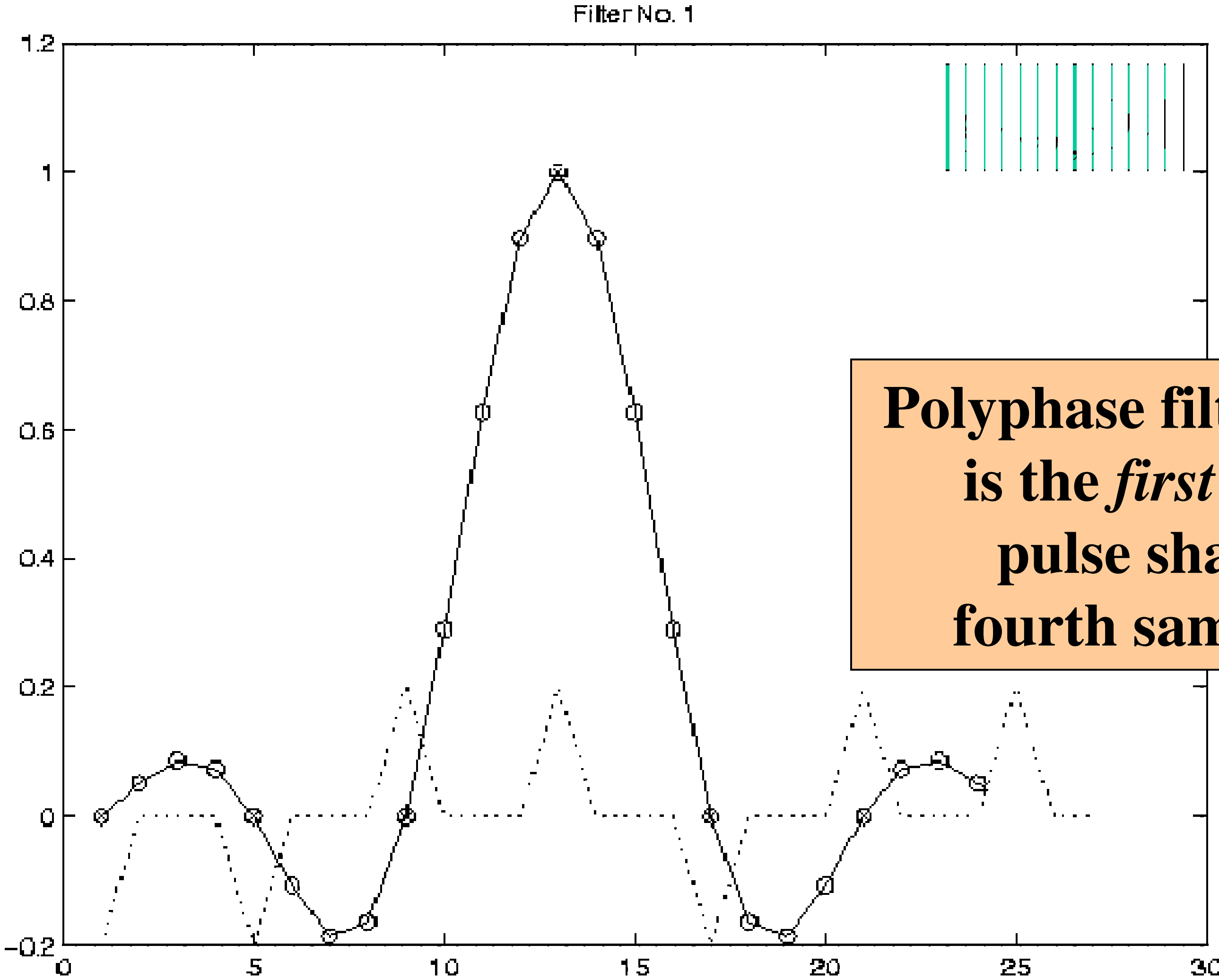
- **Define mth polyphase filter**

$$g_{T_{sym},m}[n] = g_{T_{sym}}\left(nT_{sym} + \frac{m}{L}T_{sym}\right) \qquad m = 0, 1, \ldots, L-1$$

- **Four six-tap polyphase filters (next slide)**

$$s^*\left(nT_{sym} + \frac{m}{L}T_{sym}\right) = \sum_{k=n-2}^{n+3} a_k \; g_{T_{sym},m}[n-k]$$

Pulse Shaping Filter Bank Example



24 samples in pulse

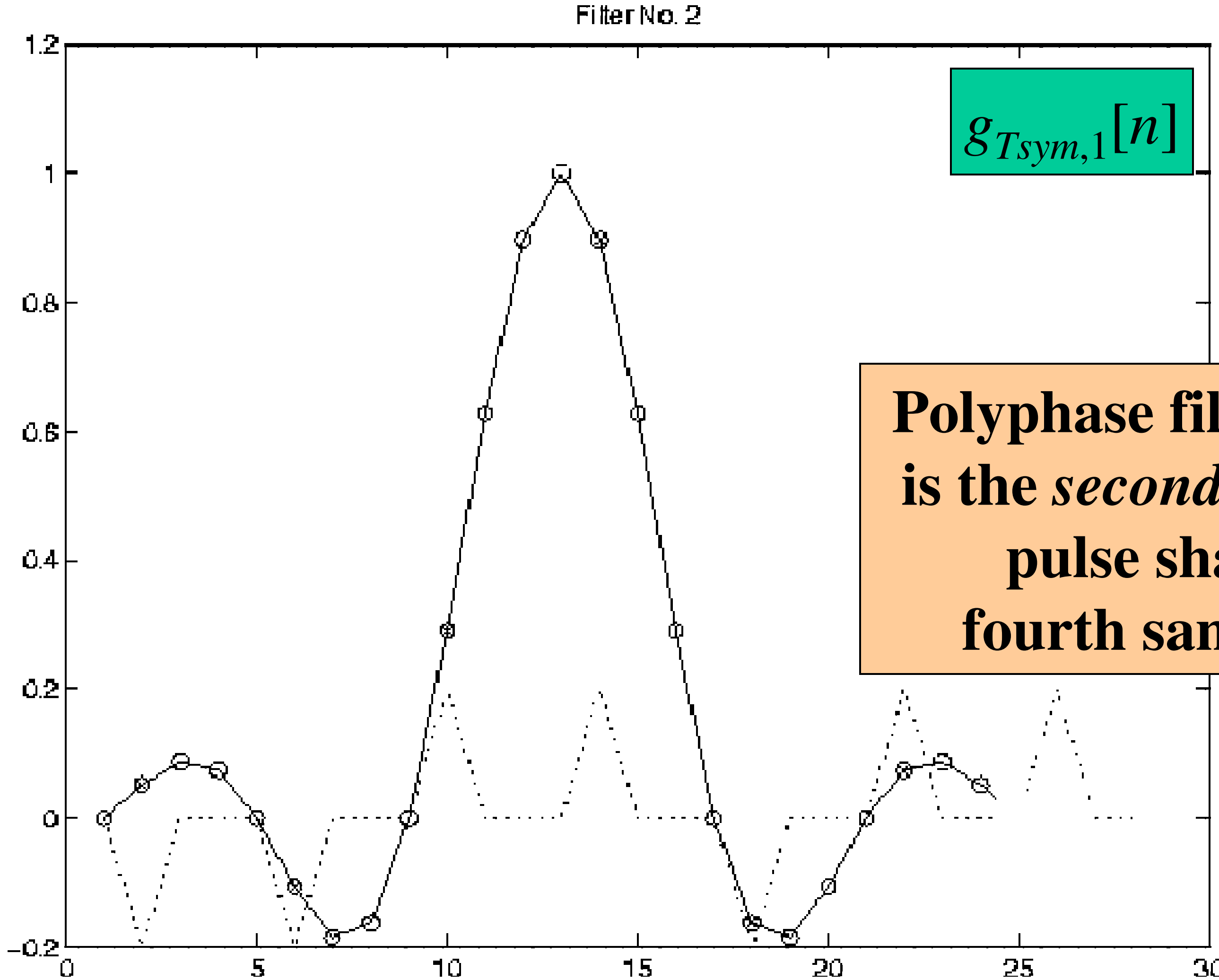
4 samples per symbol

Polyphase filter 0 response is the *first* sample of the pulse shape plus every fourth sample after that

x marks samples of polyphase filter

Polyphase filter 0 has only one non-zero sample.

Pulse Shaping Filter Bank Example



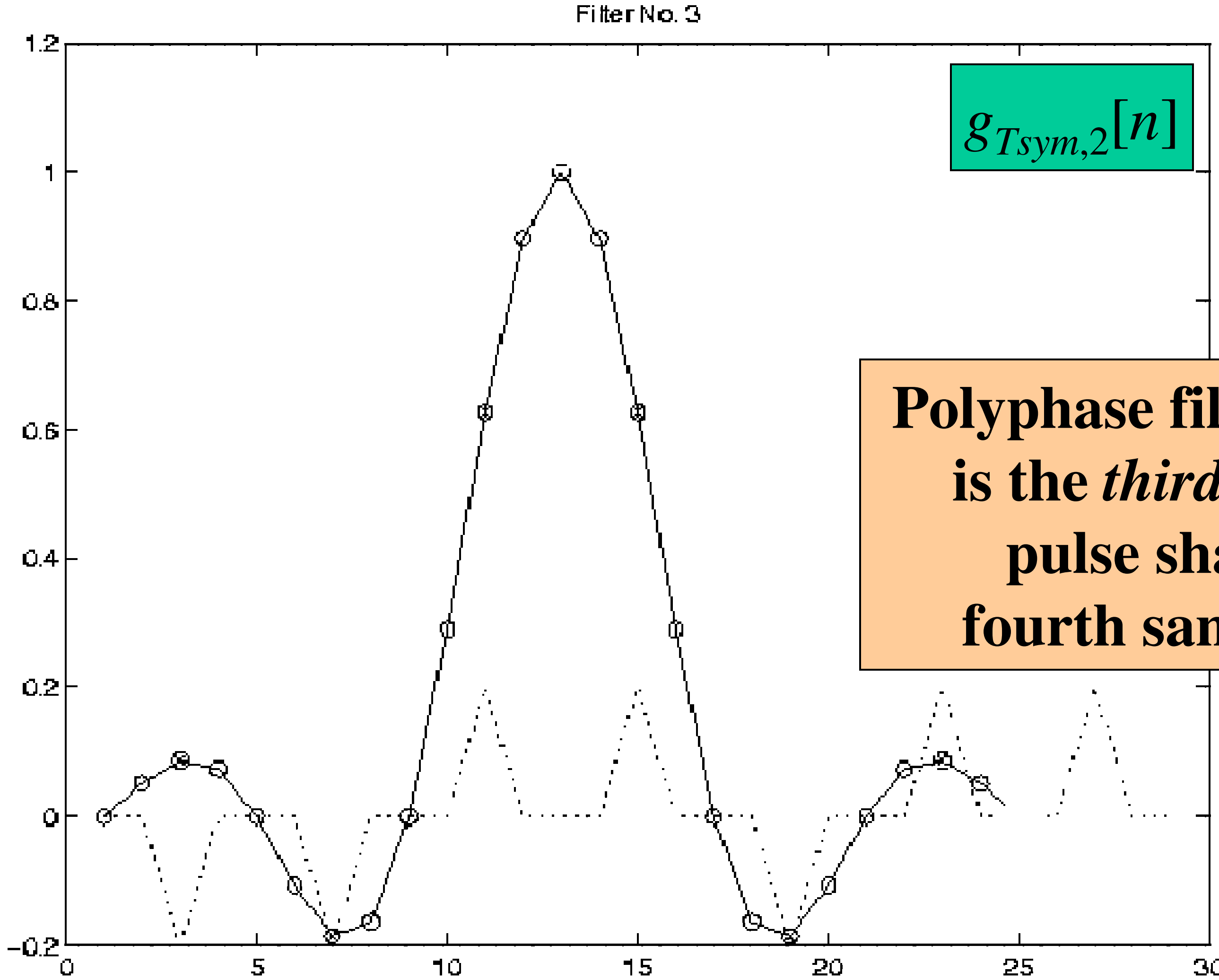
24 samples in pulse

4 samples per symbol

Polyphase filter 1 response is the *second* sample of the pulse shape plus every fourth sample after that

x marks samples of polyphase filter

Pulse Shaping Filter Bank Example



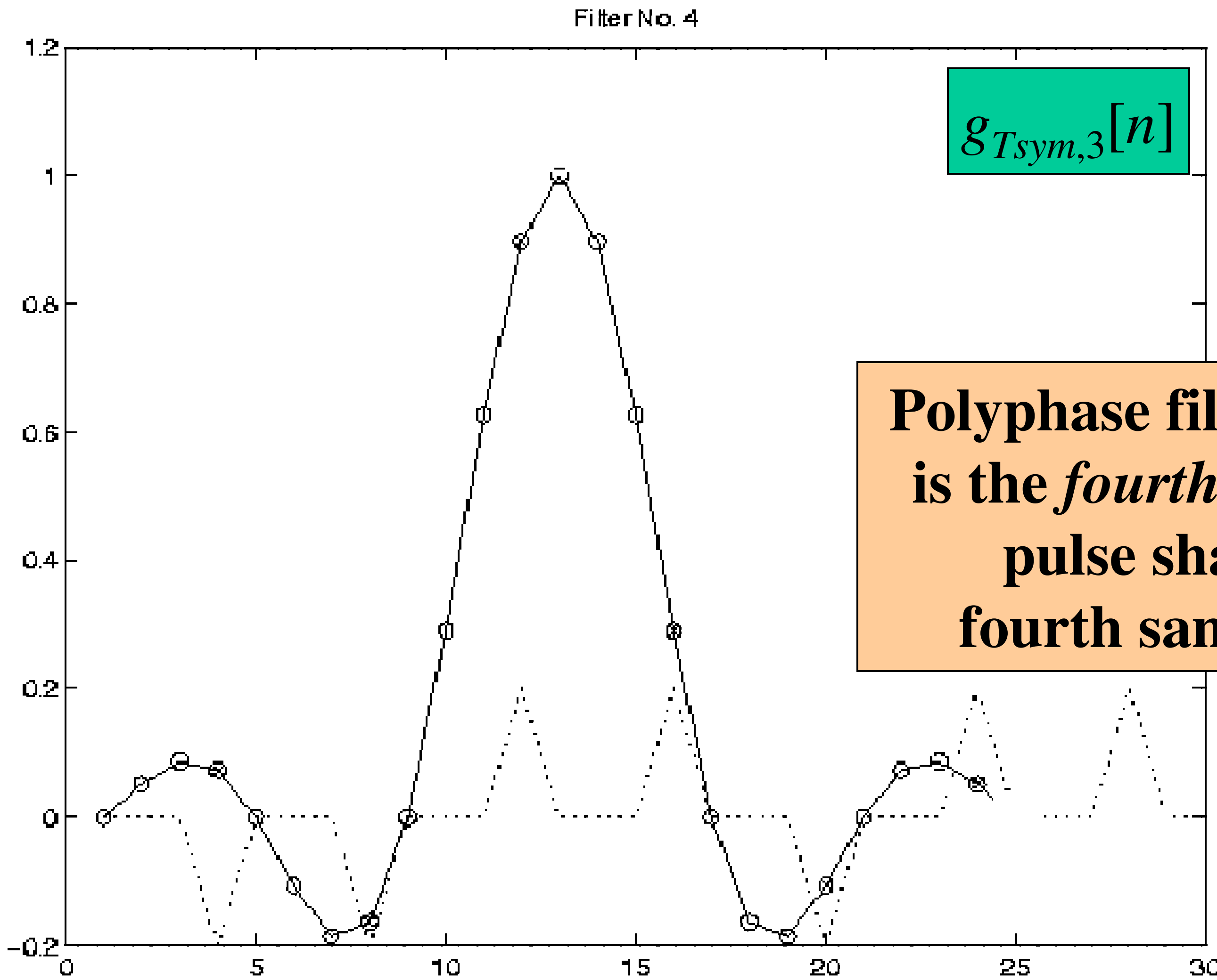
24 samples in pulse

4 samples per symbol

Polyphase filter 2 response is the *third* sample of the pulse shape plus every fourth sample after that

x marks samples of polyphase filter

Pulse Shaping Filter Bank Example



24 samples in pulse

4 samples per symbol

Polyphase filter 3 response is the *fourth* sample of the pulse shape plus every fourth sample after that

x marks samples of polyphase filter

Intersymbol Interference

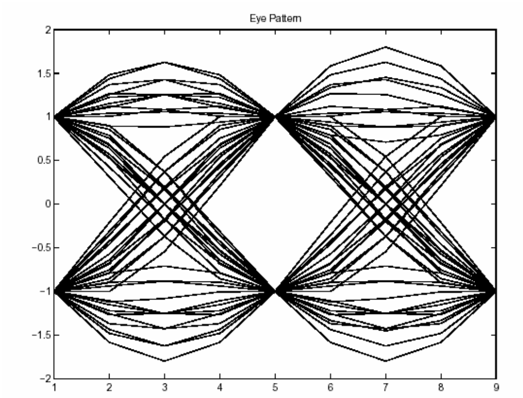
- **Eye diagram is empirical measure of quality of received signal**

$$x(n) = \sum_{k=-\infty}^{\infty} a_k \; g(nT_{sym} - k\,T_{sym}) = g(0)\left(a_n + \sum_{\substack{k=-\infty \\ k\neq n}}^{+\infty} a_k \frac{g(nT_{sym} - kT_{sym})}{g(0)} \right)$$

- **Intersymbol interference (ISI):**

$$D \leq (M-1)\, d \sum_{k=-\infty, k\neq n}^{\infty} \left| \frac{g(nT_{sym} - kT_{sym})}{g(0)} \right| = (M-1)\, d \sum_{k=-\infty, k\neq n}^{\infty} \left| \frac{g(kT_{sym})}{g(0)} \right|$$

- **Raised cosine filter has zero ISI when correctly sampled**



Optional

Symbol Clock Recovery

- **Transmitter and receiver normally have different crystal oscillators**
- **Critical for receiver to sample at correct time instances to have max signal power and min ISI**
- **Receiver should try to synchronize with transmitter clock (symbol frequency and phase)**
  - First extract clock information from received signal
  - Then either adjust analog-to-digital converter or interpolate
- **Next slides develop adjustment to A/D converter**
- **Also, see Handout M in the reader**

Optional

Symbol Clock Recovery

- **$g_1(t)$ is impulse response of LTI composite channel of pulse shaper, noise-free channel, receive filter**

$$q(t) = s^*(t) * g_1(t) = \sum_{k=-\infty}^{\infty} a_k \; g_1(t - kT_{sym})$$

$s^*(t)$ is transmitted signal

$$p(t) = q^2(t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k \; a_m \; g_1(t - kT_{sym}) \; g_1(t - mT_{sym})$$

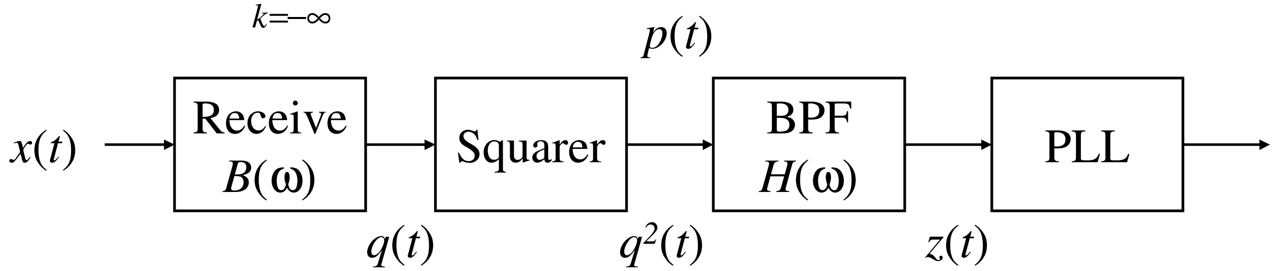
$g_1(t)$ is deterministic

$$E\{p(t)\} = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\{a_k \; a_m\} \; g_1(t - kT_{sym}) \; g_1(t - mT_{sym})$$

$E\{a_k \; a_m\} = a^2 \, \delta[k-m]$

$$= a^2 \sum_{k=-\infty}^{\infty} g_1^2(t - kT_{sym})$$

Periodic with period T_{sym}



Optional

Symbol Clock Recovery

- **Fourier series representation of E{ $p(t)$ }**

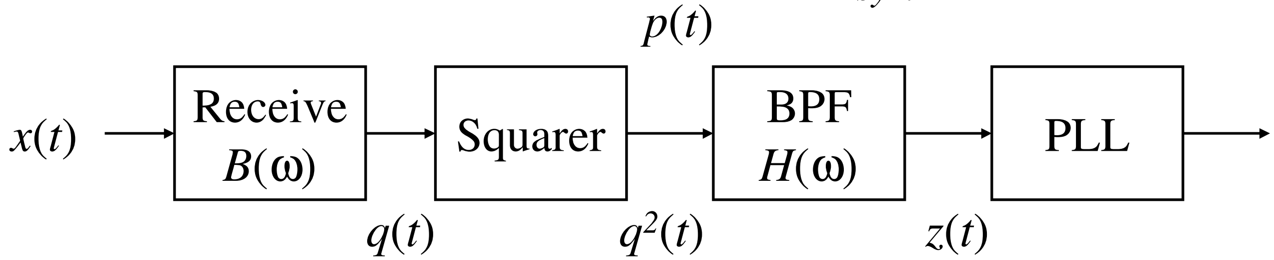
$$E\{p(t)\} = \sum_{k=-\infty}^{\infty} p_k \, e^{j\,k\,\omega_{sym}\,t} \quad \text{where} \quad p_k = \frac{1}{T_{sym}} \int_0^{T_{sym}} E\{p(t)\} e^{-j\,k\,\omega_{sym}\,t} dt$$

- **In terms of $g_1(t)$ and using Parseval's relation**

$$p_k = \frac{a^2}{T_{sym}} \int_{-\infty}^{\infty} g_1^2(t) e^{-jk\omega_{sym}t} dt = \frac{a^2}{2\pi\,T_{sym}} \int_{-\infty}^{\infty} G_1(\omega) G_1(k\omega_{sym} - \omega) d\omega$$

- **Fourier series representation of E{ $z(t)$ }**

$$z_k = p_k H(k\omega_{sym}) = H(k\omega_{sym}) \frac{a^2}{2\pi T_{sym}} \int_{-\infty}^{\infty} G_1(\omega) G_1(k\omega_{sym} - \omega) d\omega$$



Symbol Clock Recovery

- **With $G_1(\omega) = X(\omega)\, B(\omega)$**

Choose $B(\omega)$ to pass $\pm\frac{1}{2}\omega_{sym}$ → $p_k = 0$ except k = -1, 0, 1

$$Z_k = p_k H(k\omega_{sym}) = H(k\omega_{sym}) \frac{a^2}{2\pi T_{sym}} \int_{-\infty}^{\infty} G_1(\omega) G_1(k\omega_{sym} - \omega)\, d\omega$$

Choose $H(\omega)$ to pass $\pm\omega_{sym}$ → $Z_k = 0$ except k = -1, 1

$$E\{z(t)\} = \sum_k Z_k e^{jk\omega_{sym} t} = e^{-j\omega_{sym} t} + e^{j\omega_{sym} t} = 2\cos(\omega_{sym} t)$$

- **$B(\omega)$ is lowpass filter with $\omega_{passband} = \frac{1}{2}\omega_{sym}$**
- **$H(\omega)$ is bandpass filter with center frequency ω_{sym}**