Digital Pulse Amplitude Modulation

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Introduction

• Modulate \( M = 2^J \) discrete messages or \( J \) bits of information into amplitude of signal

• If amplitude mapping changes at symbol rate of \( f_{sym} \), then bit rate is \( J f_{sym} \)

• Conventional mapping of discrete messages to \( M \) uniformly space amplitudes

\[
a_i = d (2i - 1) \quad i = \frac{M}{2} + 1, \ldots, 0, \ldots, \frac{M}{2}
\]

• Pulse amplitude modulated (PAM) signal

\[
s^* (t) = \sum_{k=-\infty}^{\infty} a_k \delta (t - k T_{sym})
\]

No pulses overlap in time: requires infinite bandwidth

\[
f_{sym} = 1 / T_{sym}
\]
Pulse Shaping

- Infinite bandwidth cannot be sent in practice
- Limit bandwidth by pulse shaping filter with impulse response \( g_T(t) \)

\[
s^*(t) = \sum_{k=-\infty}^{\infty} a_k \ g_{T_{sym}}(t - kT_{sym})
\]

- \( L \) samples per symbol duration

\[
s^*(nT_{sym} + \frac{m}{L}T_{sym}) = \sum_{k=-\infty}^{\infty} a_k \ g_{T_{sym}}(nT_{sym} + \frac{m}{L}T_{sym} - kT_{sym})
\]

\( m \) is sample index in a symbol: \( m = 0, 1, 2, \ldots, L-1 \).

\[
s^*(Ln+m) = \sum_{k=-\infty}^{\infty} a_k \ g_{T_{sym}}(Ln+m-Lk) = \sum_{k=-\infty}^{\infty} a_k \ g_{T_{sym}}(L(n-k)+m)
\]
Pulse Shaping Example

2-PAM with Raised Cosine Pulse Shaping
Pulse Shaping Block Diagram

- **Upsampling by** $L$ **denoted as** $\uparrow L$
  
  Output input sample followed by $L-1$ zeros
  
  Upsampling by converts symbol rate to sampling rate

- **Pulse shaping (FIR) filter** $g_{T_{sym}}[m]$
  
  Fills in zero values generated by upsampler
  
  Multiplies by zero most of time ($L-1$ out of every $L$ times)
**Digital Interpolation Example**

- **Upsampling by 4 (denoted by $\uparrow 4$)**
  - Output input sample followed by 3 zeros
  - Four times the samples on output as input
  - Increases sampling rate by factor of 4

- **FIR filter performs interpolation**
  - Lowpass filter with stopband frequency $\omega_{\text{stopband}} \leq \pi / 4$
  - For $f_{\text{sampling}} = 176.4$ kHz, $\omega = \pi / 4$ corresponds to 22.05 kHz
Pulse Shaping Filter Bank

- **Simplify by avoiding multiplication by zero**

  Split the long pulse shaping filter into $L$ short polyphase filters operating at symbol rate.

  $a_k$ \rightarrow \uparrow L \rightarrow g_{T_{sym}}[m] \rightarrow \text{D/A} \rightarrow \text{Transmit Filter}$

  $s(Ln) \rightarrow g_{T_{sym},0}[n] \rightarrow \text{D/A} \rightarrow \text{Transmit Filter}$

  $s(Ln+1) \rightarrow g_{T_{sym},1}[n] \rightarrow \text{D/A} \rightarrow \text{Transmit Filter}$

  $\ldots$

  $s(Ln+(L-1)) \rightarrow g_{T_{sym},L-1}[n] \rightarrow \text{D/A} \rightarrow \text{Transmit Filter}$

Filter Bank Implementation
Pulse Shaping Filter Bank Example

• Pulse length 24 samples and $L = 4$ samples/symbol

$$s^*(Ln+m) = \sum_{k=n-2}^{n+3} a_k g_T(L(n-k)+m)$$

• Derivation in Tretter's manual,

$$s^*\left(nT_{sym} + \frac{m}{L}T_{sym}\right) = \sum_{k=n-2}^{n+3} a_k g_T\left(nT_{sym} + \frac{m}{L}T_{sym} - kT_{sym}\right) \quad m=0,1,...,L-1$$

• Define $m$th polyphase filter

$$g_{T_{sym},m}[n] = g_{T_{sym}}\left(nT_{sym} + \frac{m}{L}T_{sym}\right) \quad m=0,1,...,L-1$$

• Four six-tap polyphase filters (next slide)

$$s^*\left(nT_{sym} + \frac{m}{L}T_{sym}\right) = \sum_{k=n-2}^{n+3} a_k g_{T_{sym},m}[n-k]$$

Six pulses contribute to each output sample
Pulse Shaping Filter Bank Example

Polyphase filter 0 has only one non-zero sample.

Polyphase filter 0 response is the first sample of the pulse shape plus every fourth sample after that.

- 24 samples in pulse
- 4 samples per symbol
- x marks samples of polyphase filter
Pulse Shaping Filter Bank Example

Polyphase filter 1 response is the second sample of the pulse shape plus every fourth sample after that.

$g_{T_{sym},1}[n]$

24 samples in pulse

4 samples per symbol

x marks samples of polyphase filter
Pulse Shaping Filter Bank Example

$g_{T_{sym},2}[n]$  

24 samples in pulse  

4 samples per symbol  

Polyphase filter 2 response is the *third* sample of the pulse shape plus every fourth sample after that  

x marks samples of polyphase filter
Pulse Shaping Filter Bank Example

Polyphase filter 3 response is the *fourth* sample of the pulse shape plus every fourth sample after that.

$g_{Tsym,3}[n]$ [24 samples in pulse]

4 samples per symbol

x marks samples of polyphase filter
Intersymbol Interference

- Eye diagram is an empirical measure of the quality of the received signal.

\[ x(n) = \sum_{k=-\infty}^{\infty} a_k g(nT_{sym} - kT_{sym}) = g(0) \left( a_n + \sum_{k=-\infty \atop k \neq n}^{+\infty} a_k \frac{g(nT_{sym} - kT_{sym})}{g(0)} \right) \]

- Intersymbol interference (ISI):

\[ D \leq (M - 1) d \sum_{k=-\infty, k \neq n}^{\infty} \left| \frac{g(nT_{sym} - kT_{sym})}{g(0)} \right| = (M - 1) d \sum_{k=-\infty, k \neq n}^{\infty} \left| \frac{g(kT_{sym})}{g(0)} \right| \]

- Raised cosine filter has zero ISI when correctly sampled.
Symbol Clock Recovery

- Transmitter and receiver normally have different crystal oscillators
- Critical for receiver to sample at correct time instances to have max signal power and min ISI
- Receiver should try to synchronize with transmitter clock (symbol frequency and phase)
  First extract clock information from received signal
  Then either adjust analog-to-digital converter or interpolate
- Next slides develop adjustment to A/D converter
- Also, see Handout M in the reader
Symbol Clock Recovery

- $g_1(t)$ is impulse response of LTI composite channel of pulse shaper, noise-free channel, receive filter

$$q(t) = s^*(t) * g_1(t) = \sum_{k=-\infty}^{\infty} a_k \ g_1(t - kT_{\text{sym}})$$

$s^*(t)$ is transmitted signal

$$p(t) = q^2(t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k \ a_m \ g_1(t - kT_{\text{sym}}) \ g_1(t - mT_{\text{sym}})$$

$g_1(t)$ is deterministic

$$E\{p(t)\} = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\{a_k \ a_m\} \ g_1(t - kT_{\text{sym}}) \ g_1(t - mT_{\text{sym}})$$

$E\{a_k \ a_m\} = a^2 \ \delta[k-m]$  

Periodic with period $T_{\text{sym}}$

$$E\{p(t)\} = a^2 \ \sum_{k=-\infty}^{\infty} g_1^2(t - kT_{\text{sym}})$$

$g_1(t)$ is deterministic

Optional

Receive $B(\omega)$  
Squarer  
BPF $H(\omega)$  
PLL

$x(t)$  
$p(t)$  
$q(t)$  
$q^2(t)$  
z(t)
Symbol Clock Recovery

- Fourier series representation of $E\{ p(t) \}$

$$E\{ p(t) \} = \sum_{k=-\infty}^{\infty} p_k e^{j k \omega_{sym} t} \quad \text{where} \quad p_k = \frac{1}{T_{sym}} \int_{0}^{T_{sym}} E\{ p(t) \} e^{-j k \omega_{sym} t} dt$$

- In terms of $g_1(t)$ and using Parseval’s relation

$$p_k = \frac{a^2}{T_{sym}} \int_{-\infty}^{\infty} g_1^2(t) e^{-j k \omega_{sym} t} dt = \frac{a^2}{2\pi T_{sym}} \int_{-\infty}^{\infty} G_1(\omega)G_1(k\omega_{sym} - \omega) d\omega$$

- Fourier series representation of $E\{ z(t) \}$

$$z_k = p_k H(k\omega_{sym}) = H(k\omega_{sym}) \frac{a^2}{2\pi T_{sym}} \int_{-\infty}^{\infty} G_1(\omega)G_1(k\omega_{sym} - \omega) d\omega$$

\[
\begin{array}{c}
\text{x(t)} \quad \rightarrow \quad \text{Receive} \quad B(\omega) \quad \rightarrow \quad \text{Squarer} \quad \rightarrow \quad \text{BPF} \quad H(\omega) \quad \rightarrow \quad \text{PLL} \quad \rightarrow \\
\quad \rightarrow \quad q(t) \quad \rightarrow \quad q^2(t) \quad \rightarrow \quad z(t)
\end{array}
\]
Symbol Clock Recovery

- With $G_1(\omega) = X(\omega) B(\omega)$

Choose $B(\omega)$ to pass $\pm \frac{1}{2}\omega_{sym}$ $\Rightarrow p_k = 0$ except $k = -1, 0, 1$

$$Z_k = p_k H(k\omega_{sym}) = H(k\omega_{sym}) \frac{a^2}{2\pi T_{sym}} \int_{-\infty}^{\infty} G_1(\omega)G_1(k\omega_{sym}-\omega)d\omega$$

Choose $H(\omega)$ to pass $\pm\omega_{sym}$ $\Rightarrow Z_k = 0$ except $k = -1, 1$

$$E\{z(t)\} = \sum_k Z_k e^{j k\omega_{sym}t} = e^{-j\omega_{sym}t} + e^{j\omega_{sym}t} = 2\cos(\omega_{sym}t)$$

- $B(\omega)$ is lowpass filter with $\omega_{passband} = \frac{1}{2}\omega_{sym}$

- $H(\omega)$ is bandpass filter with center frequency $\omega_{sym}$