Matched Filtering and Digital Pulse Amplitude Modulation (PAM)

- **Slides by Prof. Brian L. Evans and Dr. Serene Banerjee**
- **Dept. of Electrical and Computer Engineering**
- **The University of Texas at Austin**

Lecture 13

Outline

Part I

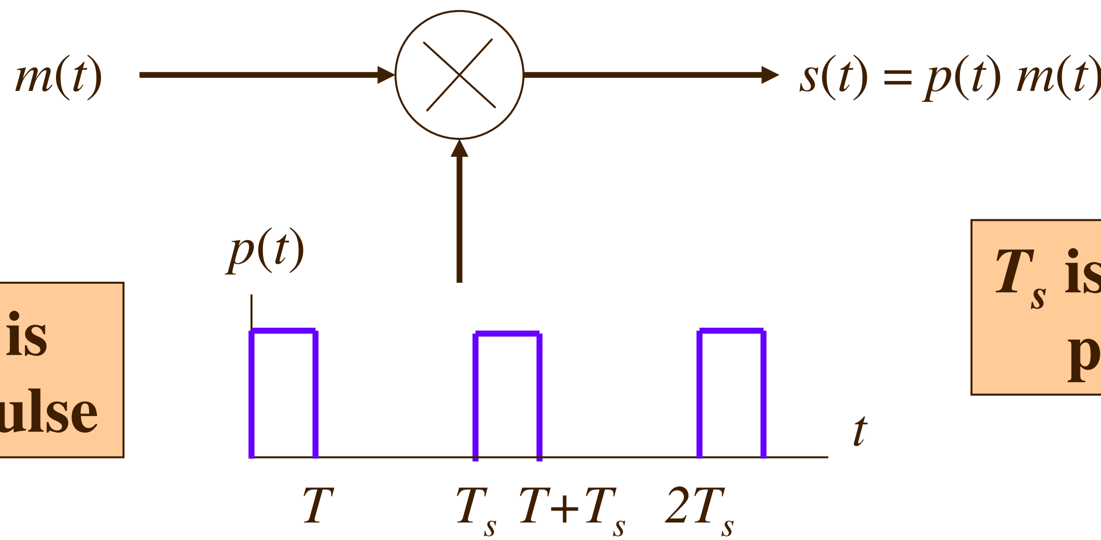
- **PAM**
- **Matched Filtering**
- **PAM System**
- **Transmit Bits**

Part II

- **Intersymbol Interference (ISI)**
  - Bit error probability for binary signals
  - Bit error probability for *M*-ary (multilevel) signals
- **Eye Diagram**

Pulse Amplitude Modulation (PAM)

- **Amplitude of periodic pulse train is varied with a sampled message signal *m***
  - **Digital PAM**: coded pulses of the sampled and quantized message signal are transmitted (next slide)
  - **Analog PAM**: periodic pulse train with period T_s is the carrier (below)

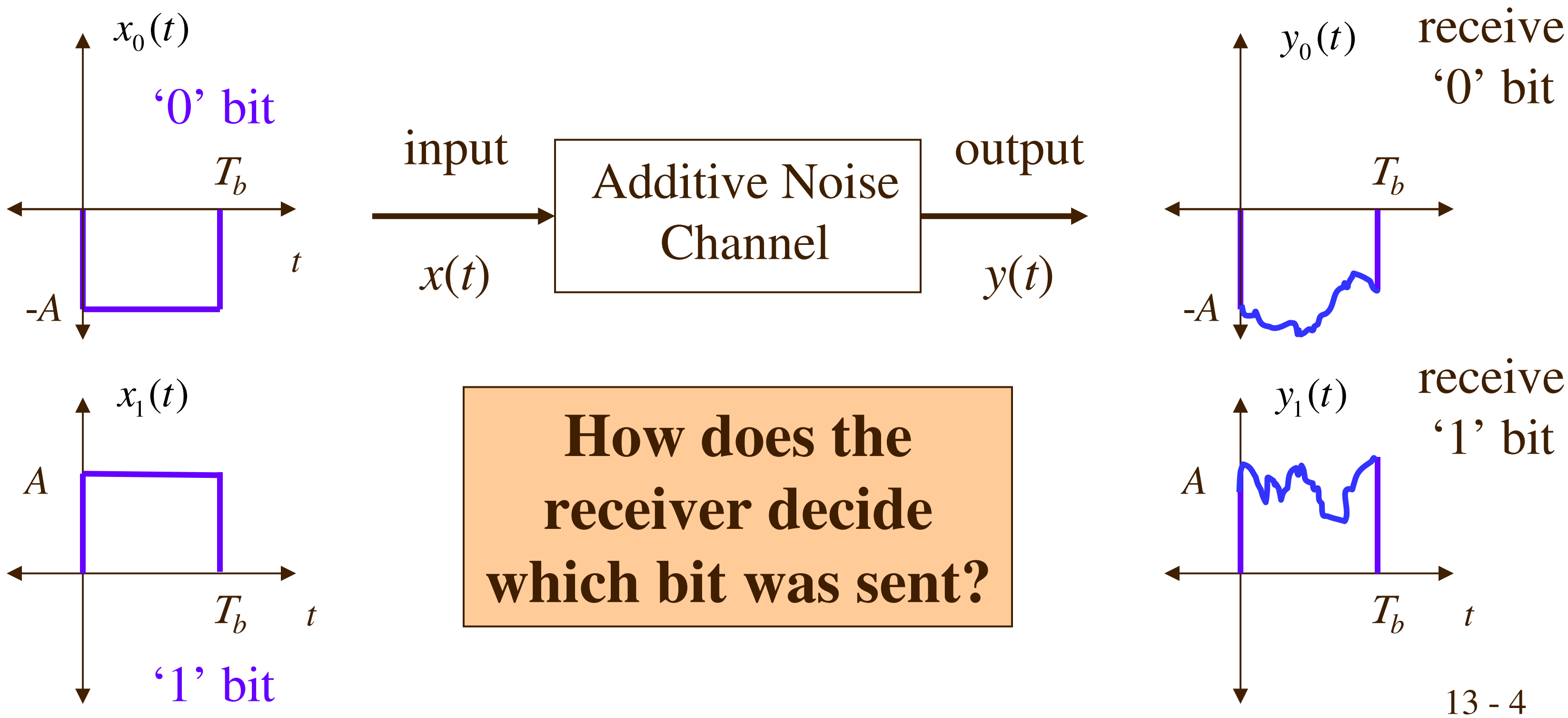


Pulse shape is rectangular pulse

T_s is symbol period

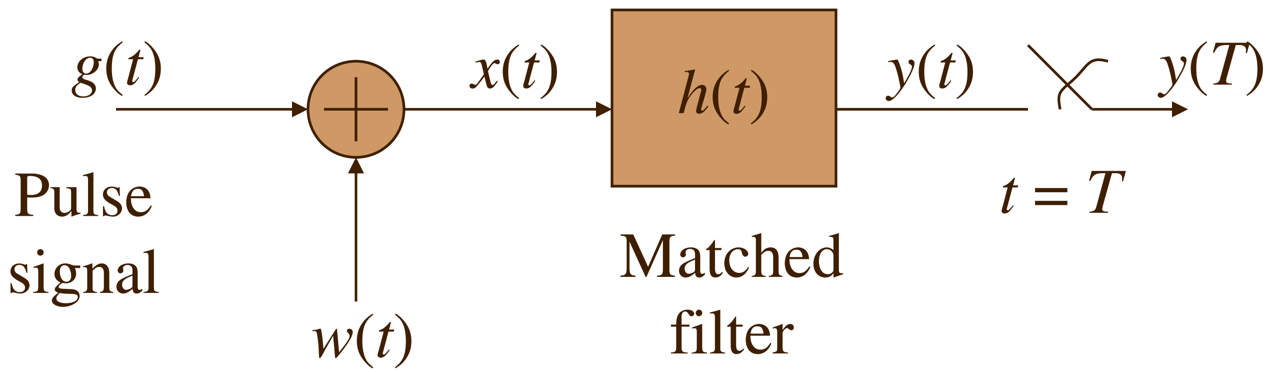
Pulse Amplitude Modulation (PAM)

- **Transmission on communication channels is analog**
- **One way to transmit digital information is called 2-level digital PAM**



Matched Filter

- **Detection of pulse in presence of additive noise**
  - Receiver knows what pulse shape it is looking for
  - Channel memory ignored (assumed compensated by other means, e.g. channel equalizer in receiver)



***T* is pulse period**

Additive white Gaussian noise (AWGN) with zero mean and variance $N_0/2$

$$y(t) = g(t) * h(t) + w(t) * h(t)$$
$$= g_0(t) + n(t)$$

Matched Filter Derivation

- **Design of matched filter**

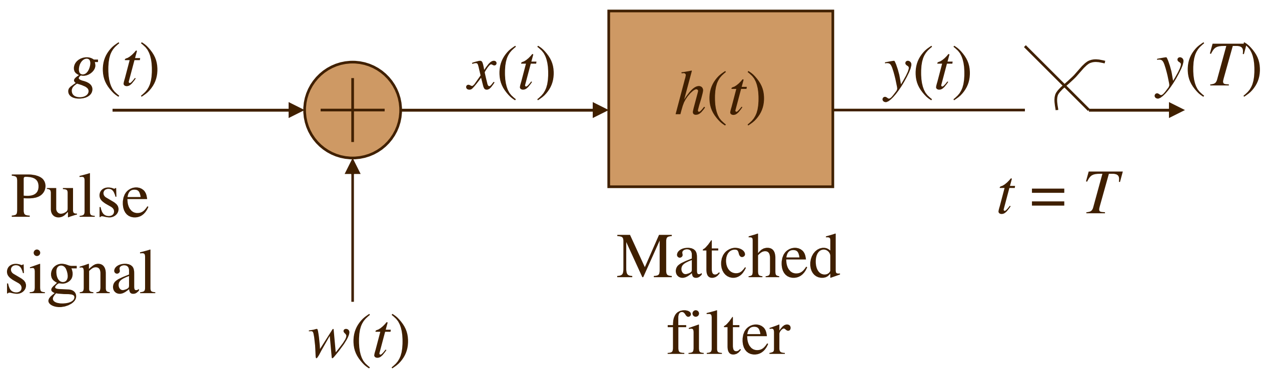
  Maximize signal power i.e. power of $g_0(t) = g(t) * h(t)$ at $t = T$

  Minimize noise i.e. power of $n(t) = w(t) * h(t)$

- **Combine design criteria**

  max η, where η is peak pulse SNR

  $$\eta = \frac{|\, g_0(T)\,|^2}{E\{n^2(t)\}} = \frac{\text{instantaneous power}}{\text{average power}}$$



Power Spectra

- **Deterministic signal $x(t)$ w/ Fourier transform $X(f)$**

  Power spectrum is square of absolute value of magnitude response (phase is ignored)

  $$P_x(f) = |X(f)|^2 = X(f)\, X^*(f)$$

  Multiplication in Fourier domain is convolution in time domain

  Conjugation in Fourier domain is reversal and conjugation in time

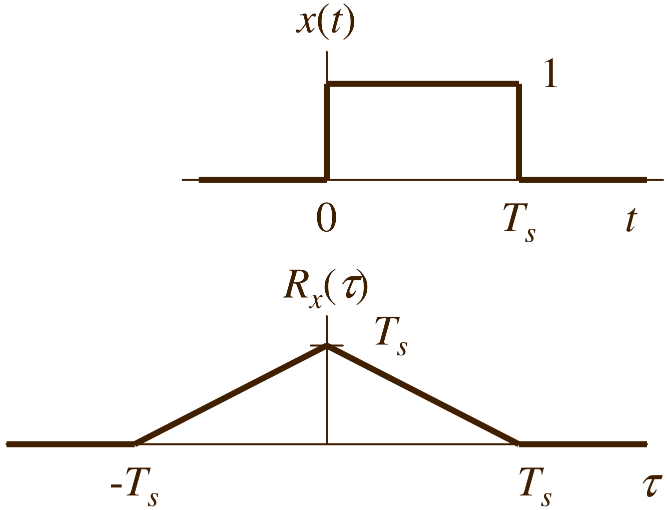
  $$X(f)\, X^*(f) = F\{ x(\tau) * x^*(-\tau) \}$$

- **Autocorrelation of $x(t)$**

  $$R_x(\tau) = x(\tau) * x^*(-\tau)$$

  Maximum value at $R_x(0)$

  $R_x(\tau)$ is even symmetric, i.e. $R_x(\tau) = R_x(-\tau)$



Power Spectra

- **Power spectrum for signal $x(t)$ is** $P_x(f) = F\{\, R_x(\tau)\, \}$

  Autocorrelation of random signal $n(t)$

$$R_n(\tau) = E\left\{ n(t)\, n^*(t+\tau) \right\} = \int_{-\infty}^{\infty} n(t)\, n^*(t+\tau)\, dt$$

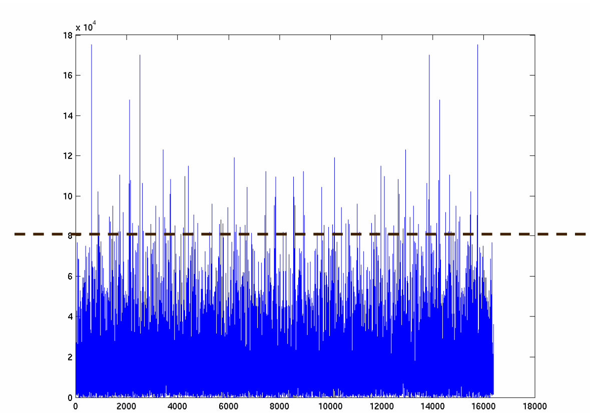
$$R_n(-\tau) = E\left\{ n(t)\, n^*(t-\tau) \right\} = \int_{-\infty}^{\infty} n(t)\, n^*(t-\tau)\, dt = n(\tau) * n^*(-\tau)$$

  For zero-mean Gaussian $n(t)$ with variance σ^2

$$R_n(\tau) = E\left\{ n(t)\, n^*(t+\tau) \right\} = \sigma^2\, \delta(\tau) \;\Leftrightarrow\; P_n(f) = \sigma^2$$

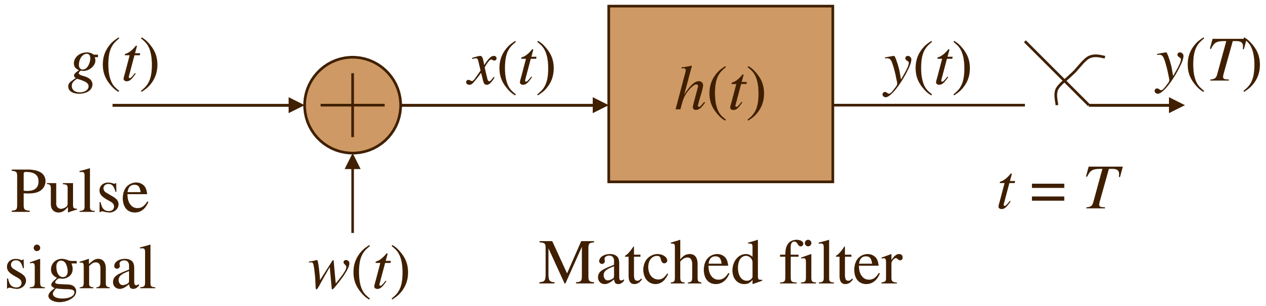
- **Estimate noise power spectrum in Matlab**

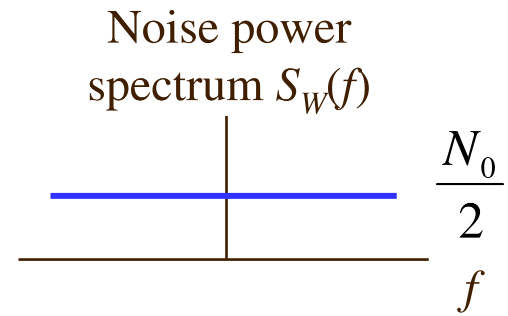
```
N = 16384;    % number of samples
gaussianNoise = randn(N,1);
plot( abs(fft(gaussianNoise)) .^ 2 );
```



noise floor

Matched Filter Derivation





- **Noise** $n(t) = w(t) * h(t)$ $\quad S_N(f) = \underbrace{S_W(f)}_{AWGN} \underbrace{S_H(f)}_{\text{Matched filter}} = \frac{N_0}{2} |H(f)|^2$

$$E\{n^2(t)\} = \int_{-\infty}^{\infty} S_N(f)\, df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df$$

- **Signal** $g_0(t) = g(t) * h(t)$ $\quad G_0(f) = H(f)G(f)$

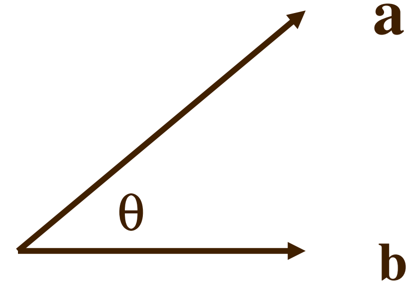
$$g_0(t) = \int_{-\infty}^{\infty} H(f)\, G(f)\, e^{j 2\pi f t}\, df$$

$$|g_0(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)\, G(f)\, e^{j 2\pi f T}\, df \right|^2$$

Matched Filter Derivation

- **Find $h(t)$ that maximizes pulse peak SNR η**

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)\, G(f)\, e^{j 2 \pi f T}\, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} | H(f) |^2\, df}$$



- **Schwartz's inequality**

For vectors: $| \mathbf{a}^T\, \mathbf{b}^* | \le \| \mathbf{a} \|\, \| \mathbf{b} \| \Leftrightarrow \cos\theta = \frac{\mathbf{a}^T \mathbf{b}}{\| \mathbf{a} \|\, \| \mathbf{b} \|}$

For functions: $\left| \int_{-\infty}^{\infty} \phi_1(x)\, \phi_2^*(x)\, dx \right|^2 \le \int_{-\infty}^{\infty} \left| \phi_1(x) \right|^2 dx \int_{-\infty}^{\infty} \left| \phi_2(x) \right|^2 dx$

lower bound reached iff $\phi_1(x) = k\, \phi_2(x)\ \ \forall k \in R$

Matched Filter Derivation

Let $\phi_1(f) = H(f)$ and $\phi_2(f) = G^*(f)\, e^{-j 2\pi f T}$

$$\left| \int_{-\infty}^{\infty} H(f)\, G(f)\, e^{j 2\pi f T}\, df \right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\, df \int_{-\infty}^{\infty} |G(f)|^2\, df$$

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)\, G(f)\, e^{j 2\pi f T}\, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df} \le \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2\, df$$

$\eta_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2\, df$, which occurs when

$H_{opt}(f) = k\, G^*(f)\, e^{-j 2\pi f T} \quad \forall k$ by Schwartz's inequality

Hence, $h_{opt}(t) = k\, g^*(T - t)$

Matched Filter

- **Given transmitter pulse shape** $g(t)$ **of duration** T**, matched filter is given by** $h_{opt}(t) = k\, g^*(T-t)$ **for all** k

  Duration and shape of impulse response of the optimal filter is determined by pulse shape $g(t)$

  $h_{opt}(t)$ is scaled, time-reversed, and shifted version of $g(t)$

- **Optimal filter maximizes peak pulse SNR**

$$\eta_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 \, df = \frac{2}{N_0} \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \frac{2E_b}{N_0} = \text{SNR}$$

  Does not depend on pulse shape $g(t)$

  Proportional to signal energy (energy per bit) E_b

  Inversely proportional to power spectral density of noise

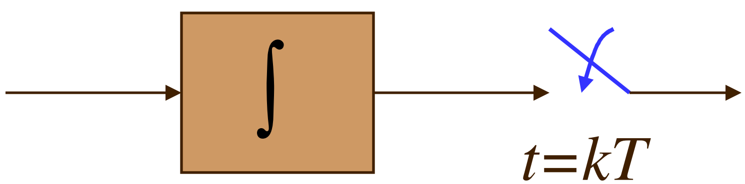
Matched Filter for Rectangular Pulse

- **Matched filter for causal rectangular pulse has an impulse response that is a causal rectangular pulse**
- **Convolve input with rectangular pulse of duration T sec and sample result at T sec is same as to**

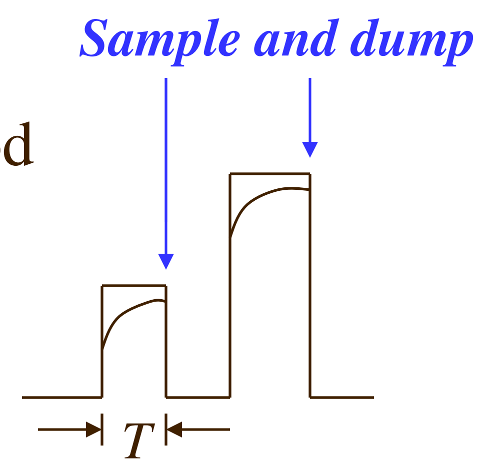
  First, integrate for T sec

  Second, sample at symbol period T sec

  Third, reset integration for next time period
- **Integrate and dump circuit**

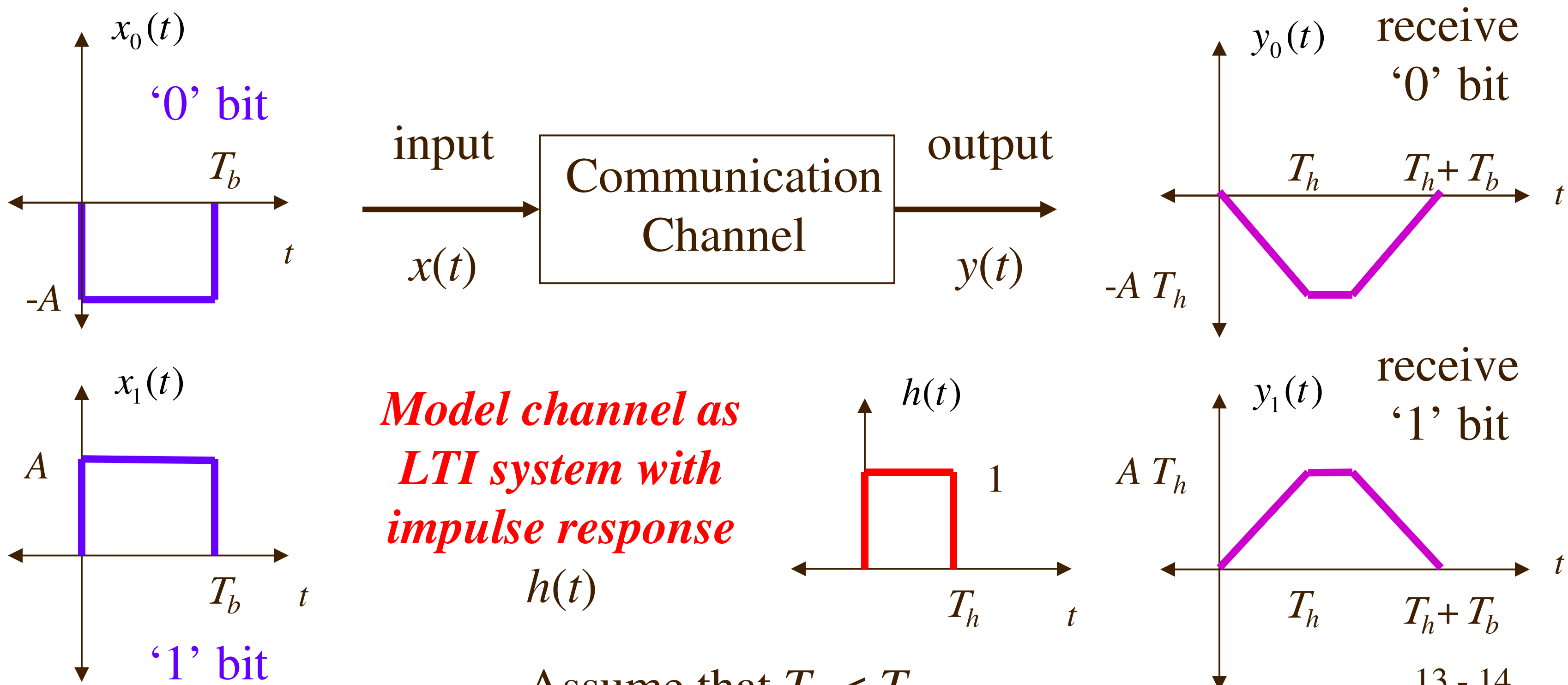


$h(t)$ = ___



Transmit One Bit

- **Analog transmission over communication channels**
- **Two-level digital PAM over channel that has memory but does not add noise**

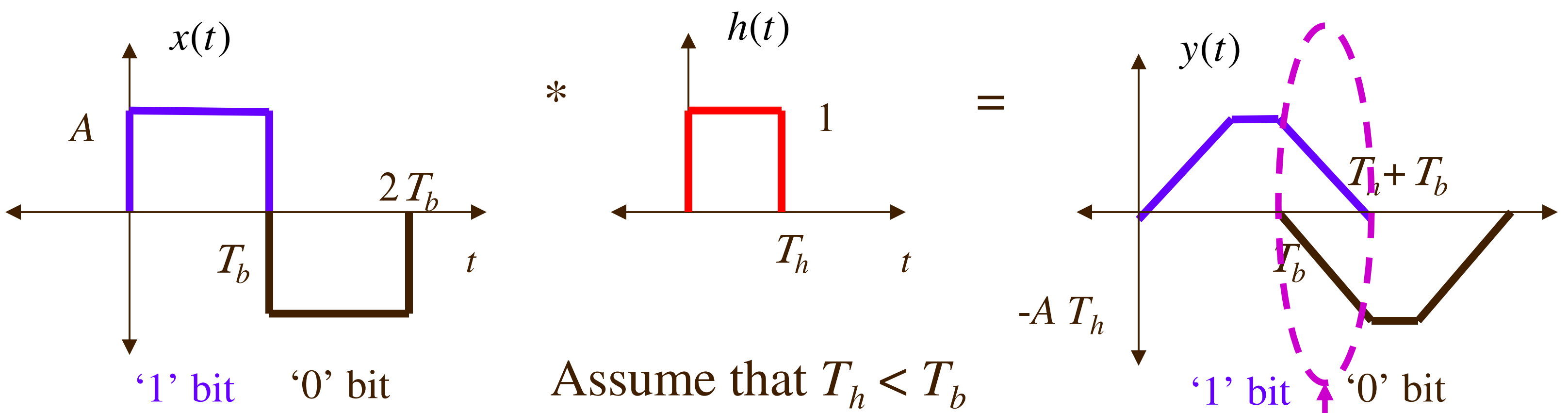


Model channel as LTI system with impulse response $h(t)$

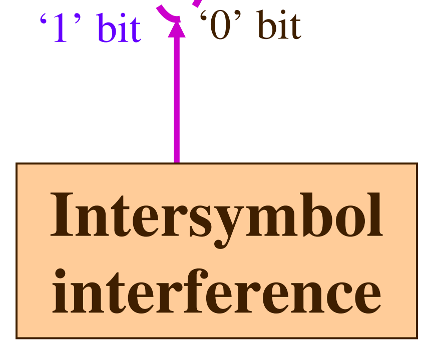
Assume that $T_h < T_b$

Transmit Two Bits (Interference)

- **Transmitting two bits (pulses) back-to-back will cause overlap (interference) at the receiver**

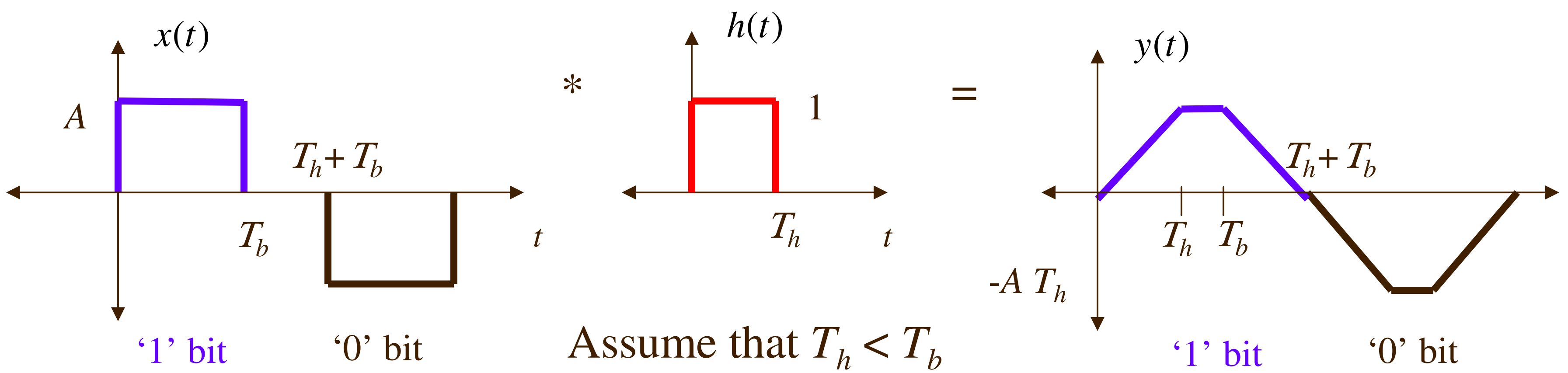


- **Sample $y(t)$ at T_b, $2\ T_b$, …, and threshold with threshold of zero**
- **How do we prevent intersymbol interference (ISI) at the receiver?**



Transmit Two Bits (No Interference)

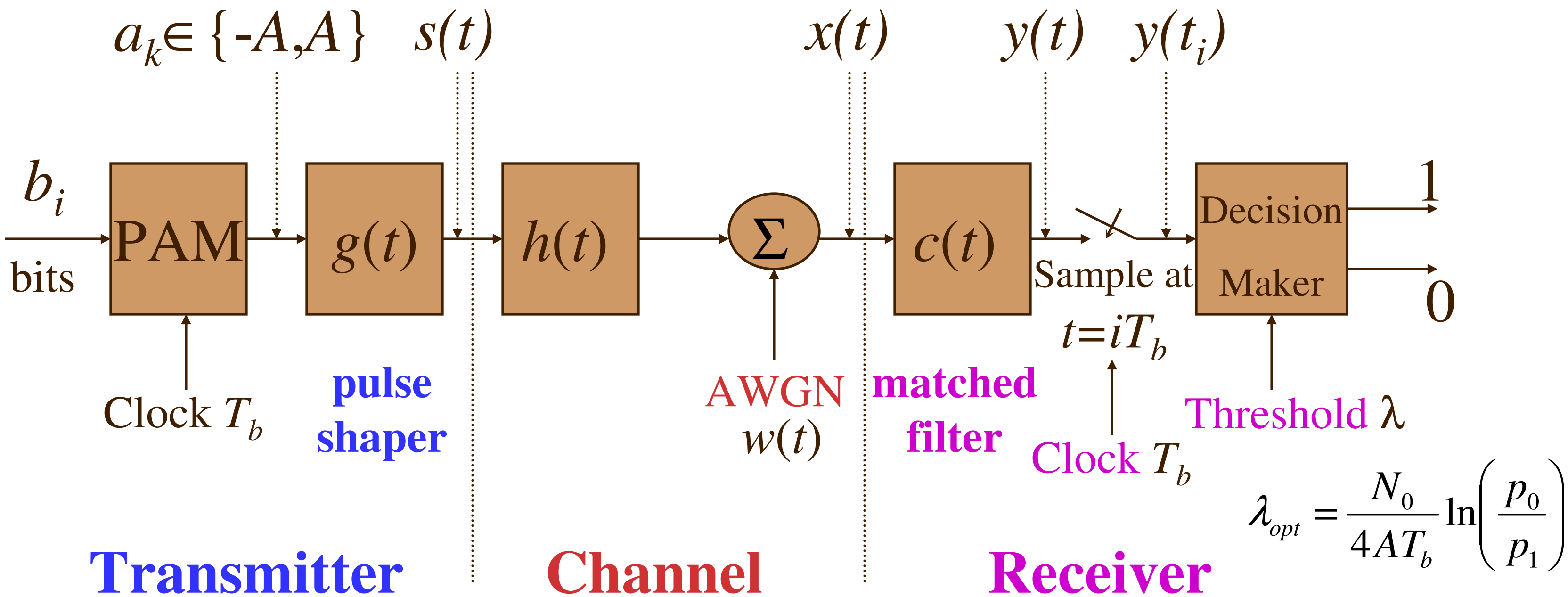
- **Prevent intersymbol interference by waiting T_h seconds between pulses (called a guard period)**



Assume that $T_h < T_b$

- **Disadvantages?**

Digital 2-level PAM System



- **Transmitted signal** $s(t) = \sum_k a_k \, g(t - k\,T_b)$
- **Requires synchronization of clocks between transmitter and receiver**

Digital PAM Receiver

- **Why is $g(t)$ a pulse and not an impulse?**

Otherwise, $s(t)$ would require infinite bandwidth

$$s(t) = \sum_k a_k \delta(t - k\,T_b)$$

Since we cannot send an signal of infinite bandwidth, we limit its bandwidth by using a pulse shaping filter

- **Neglecting noise, would like $y(t) = g(t) * h(t) * c(t)$ to be a pulse, i.e. $y(t) = \mu\, p(t)$, to eliminate ISI**

$$y(t) = \mu \sum_k a_k\, p(t - kT_b) + n(t) \text{ where } n(t) = w(t) * c(t)$$

$$\Rightarrow y(t_i) = \underbrace{\mu\, a_i\; p(t_i - iT_b)}_{\text{actual value (note that } t_i = i\,T_b)} + \underbrace{\mu \sum_{k, k \neq i} a_k\; p\big((i-k)T_b\big)}_{\text{intersymbol interference (ISI)}} + \underbrace{n(t_i)}_{\text{noise}}$$

$p(t)$ is centered at origin

Eliminating ISI in PAM

- **One choice for $P(f)$ is a rectangular pulse**
  - W is the bandwidth of the system
  - Inverse Fourier transform of a rectangular pulse is is a sinc function

    $p(t) = \text{sinc}(2\pi W t)$

$$P(f) = \begin{cases} \dfrac{1}{2W} & , -W < f < W \\ 0 & , |f| > W \end{cases}$$

$$P(f) = \frac{1}{2W} \text{rect}\left(\frac{f}{2W}\right)$$

- **This is called the Ideal Nyquist Channel**
- **It is not realizable because the pulse shape is not causal and is infinite in duration**

Eliminating ISI in PAM

- Another choice for $P(f)$ is a raised cosine spectrum

$$P(f) = \begin{cases} \dfrac{1}{2W} & 0 \le |f| < f_1 \\ \dfrac{1}{4W}\left(1 - \sin\left(\dfrac{\pi(|f| - W)}{2W - 2f_1}\right)\right) & f_1 \le |f| < 2W - f_1 \\ 0 & 2W - f_1 \le |f| \le 2W \end{cases}$$

- Roll-off factor gives bandwidth in excess of bandwidth W for ideal Nyquist channel

$$\alpha = 1 - \frac{f_1}{W}$$

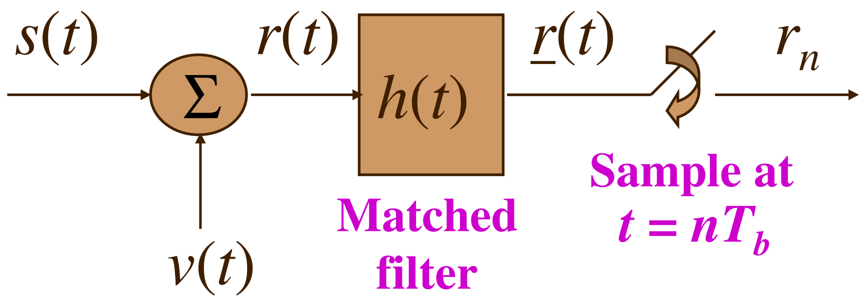
- Raised cosine pulse has zero ISI when sampled correctly

$$p(t) = \underbrace{\operatorname{sinc}\left(\frac{t}{T_s}\right)}_{\text{ideal Nyquist channel impulse response}} \underbrace{\frac{\cos(2\pi\alpha W t)}{1 - 16\alpha^2 W^2 t^2}}_{\text{dampening adjusted by rolloff factor } \alpha}$$

- Let $g(t)$ and $c(t)$ be square root raised cosines

Bit Error Probability for 2-PAM

- T_b **is bit period (bit rate is** $f_b = 1/T_b$**)**



$$s(t) = \sum_k a_k \, g(t - k\,T_b)$$

$$r(t) = s(t) + v(t)$$

$$\underline{r}(t) = h(t) * r(t)$$

$v(t)$ is AWGN with zero mean and variance σ^2

- **Lowpass filtering a Gaussian random process produces another Gaussian random process**

  Mean scaled by $H(0)$

  Variance scaled by twice lowpass filter's bandwidth

- **Matched filter's bandwidth is** $\frac{1}{2} f_b$

Bit Error Probability for 2-PAM

- **Binary waveform (rectangular pulse shape) is** $\pm A$ **over** nth **bit period** $nT_b < t < (n+1)T_b$
- **Matched filtering by integrate and dump**

  Set gain of matched filter to be $\mathbf{1/T_b}$

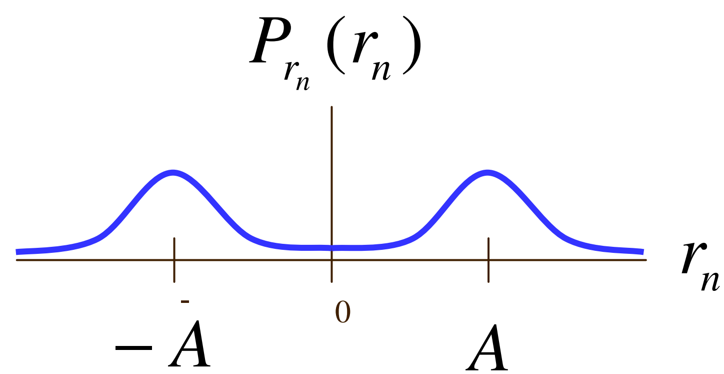
  Integrate received signal over period, scale, sample

See Slide 13-13

$$r_n = \frac{1}{T_b} \int_{nT_b}^{(n+1)T_b} r(t)\, dt$$

$$= \pm A + \frac{1}{T_b} \int_{nT_b}^{(n+1)T_b} v(t)\, dt$$

$$= \pm A + v_n$$



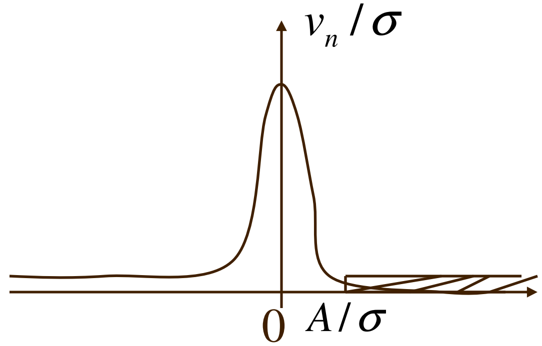
Probability density function (PDF)

Bit Error Probability for 2-PAM

- Probability of error given that the transmitted pulse has an amplitude of –A

$$P(\text{error} \mid s(nT_b) = -A) = P(-A + v_n > 0) = P(v_n > A) = P\left(\frac{v_n}{\sigma} > \frac{A}{\sigma}\right)$$

- Random variable $\frac{v_n}{\sigma}$ is Gaussian with zero mean and variance of one



PDF for $N(0, 1)$

$$P(\text{error} \mid s(nT) = -A) = P\left(\frac{v_n}{\sigma} > \frac{A}{\sigma}\right) = \int_{\frac{A}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \, dv = Q\left(\frac{A}{\sigma}\right)$$

Q function on next slide

Q Function

- **Q function**

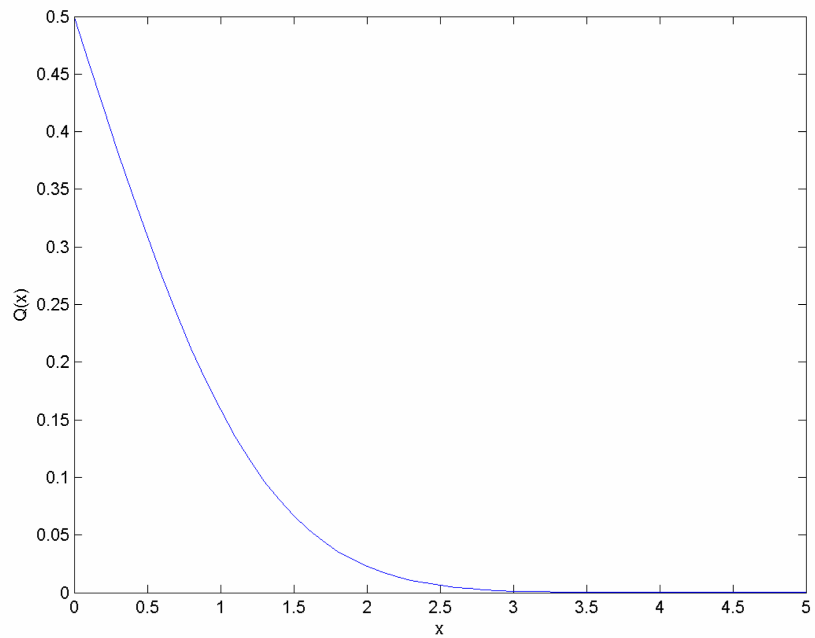
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy$$

- **Complementary error function erfc**

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

- **Relationship**

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$



Erfc[*x*] in Mathematica

erfc(*x*) in Matlab

Bit Error Probability for 2-PAM

- **Probability of error given that the transmitted pulse has an amplitude of A**

$$P(\text{error} \mid s(nT_b) = A) = Q(A/\sigma)$$

- **Assume that 0 and 1 are equally likely bits**

$$P(\text{error}) = P(A)P(\text{error} \mid s(nT_b) = A) + P(-A)P(\text{error} \mid s(nT_b) = -A)$$

$$= \frac{1}{2}Q\left(\frac{A}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\rho}\right)$$

where, $\rho = \text{SNR} = \dfrac{A^2}{\sigma^2}$

- **Probablity of error decreases exponentially with SNR**

$$erfc(x) \leq \frac{e^{-x^2}}{\sqrt{\pi}\,x} \Rightarrow Q(\sqrt{\rho}) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\rho}{2}}}{\sqrt{\rho}}$$

for large positive x, ρ

PAM Symbol Error Probability

- **Average signal power**

$$P_{Signal} = \frac{E\{a_n^2\}}{T_{sym}} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_T(\omega)|^2 \, d\omega = \frac{E\{a_n^2\}}{T_{sym}}$$

$G_T(\omega)$ is square root of the raised cosine spectrum

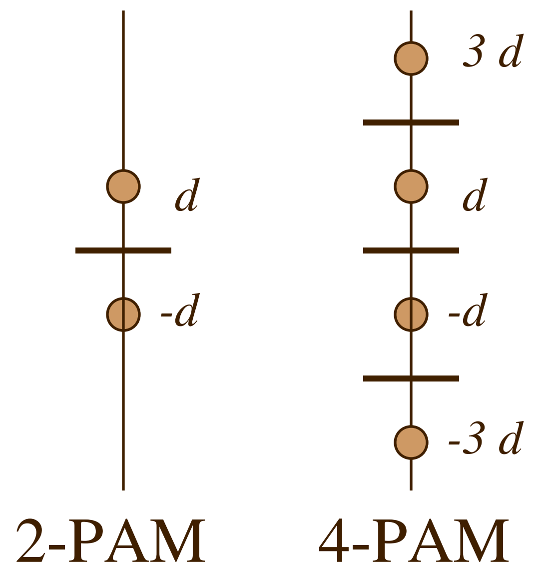
Normalization by T_{sym} will be removed in lecture 15 slides

- ***M*-level PAM amplitudes**

$$l_i = d(2i-1), \quad i = -\frac{M}{2}+1, \ldots, 0, \ldots, \frac{M}{2}$$

- **Assuming each symbol is equally likely**

$$P_{Signal} = \frac{1}{T_{sym}} \left(\sum_{i=1}^{M} l_i^2 \right) = \frac{1}{T} \left(\frac{2}{M} \sum_{i=1}^{\frac{M}{2}} [d(2i-1)]^2 \right) = (M^2-1) \frac{d^2}{3T_{sym}}$$



Constellations with decision boundaries

PAM Symbol Error Probability

- **Noise power and SNR**

$$P_{Noise} = \frac{1}{2\pi} \int_{-\omega_{sym}/2}^{\omega_{sym}/2} \underbrace{\frac{N_0}{2}}_{} \, d\omega = \frac{N_0}{2T_{sym}}$$

two-sided power spectral density of AWGN

$$\text{SNR} = \frac{P_{Signal}}{P_{Noise}} = \frac{2(M^2-1)}{3} \times \frac{d^2}{N_0}$$

- **Assume ideal channel, i.e. one without ISI**

$$x(nT_{sym}) = a_n + \underbrace{v_R(nT_{sym})}_{}$$

channel noise filtered by receiver and sampled

- **Consider M-2 inner levels in constellation**

Error if and only if

$$|v_R(nT_{sym})| > d$$

where $\sigma^2 = N_0 / 2$

Probablity of error is

$$P(|v_R(nT_{sym})| > d) = 2\,Q\left(\frac{d}{\sigma}\right)$$

- **Consider two outer levels in constellation**

$$P(v_R(nT_{sym}) > d) = Q\left(\frac{d}{\sigma}\right)$$

PAM Symbol Error Probability

- **Assuming that each symbol is equally likely, symbol error probability for *M*-level PAM**

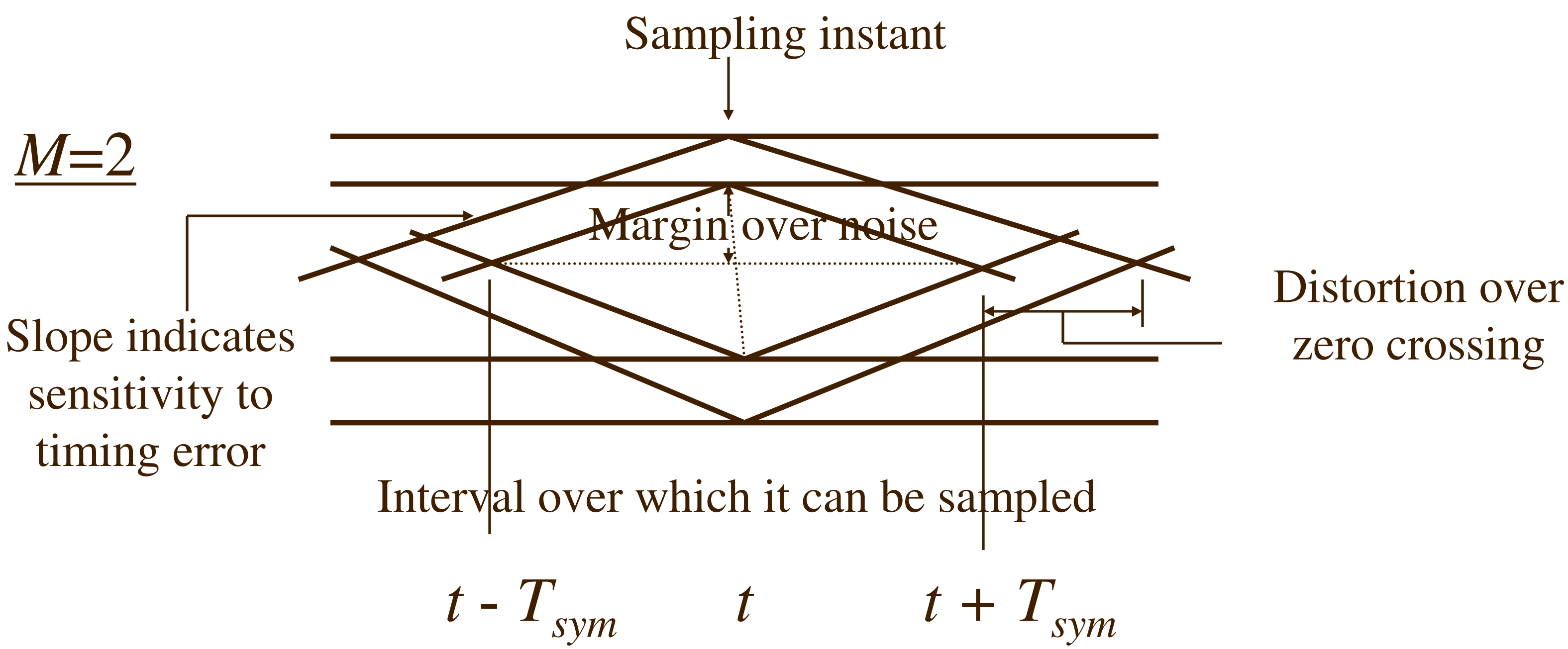
$$P_e = \underbrace{\frac{M-2}{M}\left(2\,Q\left(\frac{d}{\sigma}\right)\right)}_{M\text{-2 interior points}} + \underbrace{\frac{2}{M}Q\left(\frac{d}{\sigma}\right)}_{2\text{ exterior points}} = \frac{2(M-1)}{M}Q\left(\frac{d}{\sigma}\right)$$

- **Symbol error probability in terms of SNR**

$$P_e = 2\frac{M-1}{M}Q\left(\left(\frac{3}{M^2-1}\text{SNR}\right)^{\frac{1}{2}}\right) \quad \text{since} \quad \text{SNR} = \frac{P_{Signal}}{P_{Noise}} = \frac{d^2}{3\sigma^2}\left(M^2-1\right)$$

Eye Diagram

- **PAM receiver analysis and troubleshooting**



- **The more open the eye, the better the reception**

Eye Diagram for 4-PAM