Infinite Impulse Response Filters

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Lecture 6

Digital IIR Filters

- **Infinite Impulse Response (IIR) filter has impulse response of infinite duration, e.g.**

$$h[k]=\left(\frac{1}{2}\right)^k u[k] \quad \overset{Z}{\longleftrightarrow} \quad H(z)=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^{-k}=\frac{1}{2}+\frac{1}{4}z^{-1}+\ \ldots\ =\frac{1}{1-\frac{1}{2}z^{-1}}$$

- **How to implement the IIR filter by computer?**

Let $x[k]$ be the input signal and $y[k]$ the output signal,

$$H(z)=\frac{Y(z)}{X(z)} \Rightarrow Y(z)=H(z)X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}X(z) \Rightarrow Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)$$

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z) \Rightarrow y[k]-\frac{1}{2}y[k-1]=x[k] \Rightarrow y[k]=\frac{1}{2}y[k-1]+x[k]$$

Recursively compute output, given $y[-1]$ and $x[k]$

Different Filter Representations

- **Difference equation**

$$y[k] = \frac{1}{2} y[k-1] + \frac{1}{8} y[k-2] + x[k]$$

Recursive computation needs $y[-1]$ and $y[-2]$

For the filter to be LTI, $y[-1] = 0$ and $y[-2] = 0$

- **Transfer function**

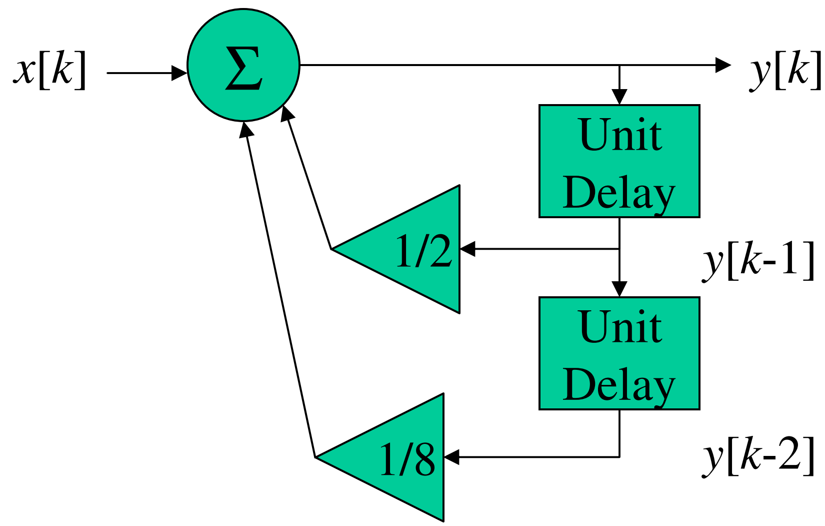
Assumes LTI system

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1} - \frac{1}{8} z^{-2}}$$

Poles at –0.183 and +0.683

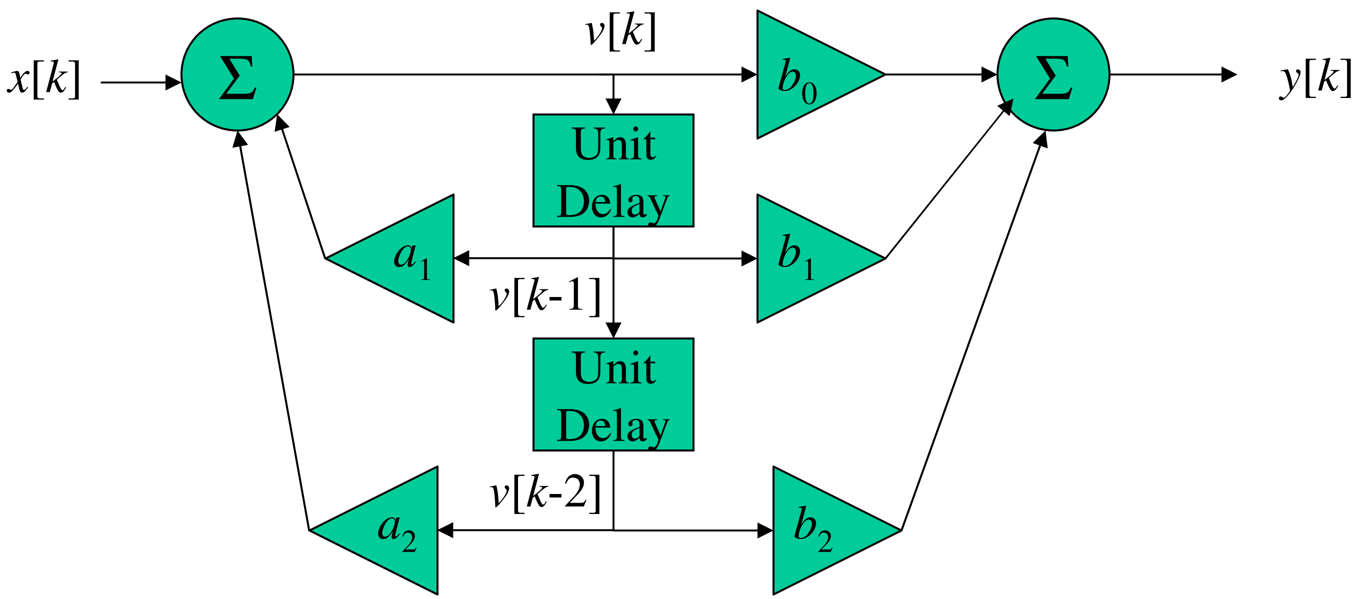
- **Block diagram representation**



Second-order filter section (a.k.a. biquad) with 2 poles and 0 zeros

Digital IIR Biquad

- Two poles and zero, one, or two zeros



- Take *z*-transform of biquad structure

$$H(z) = \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} \frac{Y(z)}{V(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

- Real coefficients a_1, a_2, b_0, b_1, and b_2 means poles and zeros in conjugate symmetric pairs $\alpha \pm j\,\beta$

Digital IIR Filter Design

- **Poles near unit circle indicate filter's passband(s)**
- **Zeros on/near unit circle indicate stopband(s)**
- **Biquad with zeros z_0 and z_1, and poles p_0 and p_1**

Transfer function $$H(z) = C \frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)}$$

Magnitude response $$\left|H(e^{j\omega})\right| = \left| C \frac{\left(e^{j\omega} - z_0\right)\left(e^{j\omega} - z_1\right)}{\left(e^{j\omega} - p_0\right)\left(e^{j\omega} - p_1\right)} \right|$$

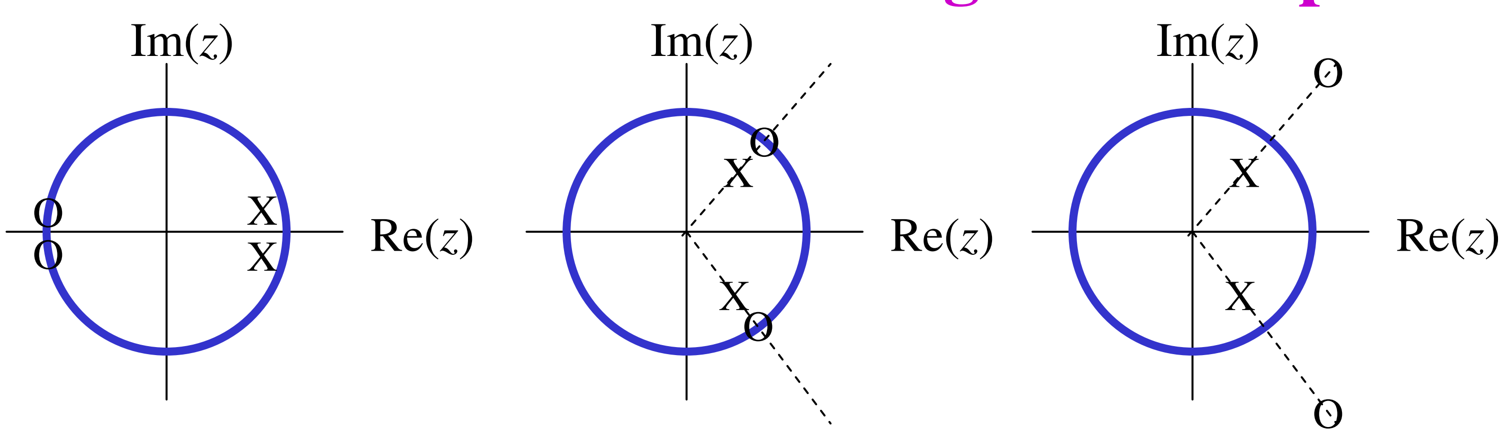
$$\left|H(e^{j\omega})\right| = |C| \frac{\left|e^{j\omega} - z_0\right| \left|e^{j\omega} - z_1\right|}{\left|e^{j\omega} - p_0\right| \left|e^{j\omega} - p_1\right|}$$

$|a - b|$ is distance between complex numbers a and b

Distance from point on unit circle $e^{j\omega}$ and pole location p_0

Digital IIR Biquad Design Examples

- **Transfer function** $H(z) = C \frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)} = C \frac{(1 - z_0 z^{-1})(1 - z_1 z^{-1})}{(1 - p_0 z^{-1})(1 - p_1 z^{-1})}$
- **Poles (X) & zeros (O) in conjugate symmetric pairs**
  – For coefficients in unfactored transfer function to be real
- **Filters below have what magnitude responses?**



lowpass
highpass
bandpass
bandstop
allpass
notch?

A Direct Form IIR Realization

- **IIR filters having rational transfer functions**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ \ldots\ + b_N z^{-N}}{1 - a_1 z^{-1} - \ \ldots\ - a_M z^{-M}} \Rightarrow Y(z)\left(1 - \sum_{m=1}^{M} a_m z^{-m}\right) = X(z) \sum_{n=0}^{N} b_n z^{-n}$$

- **Direct form realization**

$$y[k] = \sum_{n=0}^{N} b_k\, x[k-n] + \sum_{m=1}^{M} a_k\, y[k-m]$$

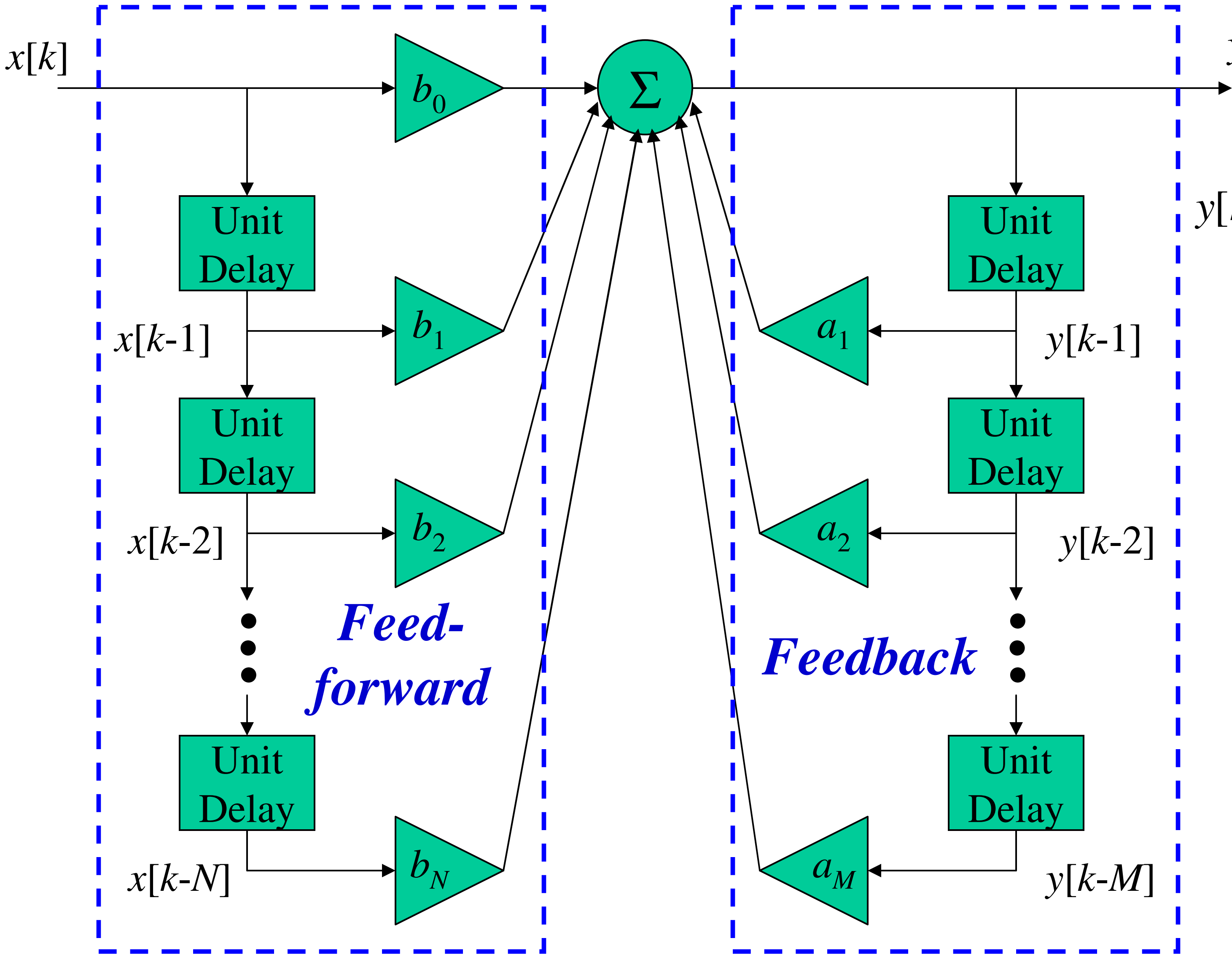
  – Dot product of vector of $N+1$ coefficients and vector of current input and previous N inputs (FIR section)

  – Dot product of vector of M coefficients and vector of previous M outputs (*FIR filtering of previous output values*)

  – Computation: $M + N + 1$ MACs

  – Memory: $M + N$ words for previous inputs/outputs and $M + N + 1$ words for coefficients

Filter Structure As a Block Diagram



$$y[k] = \sum_{n=0}^{N} b_k \; x[k-n] + \sum_{m=1}^{M} a_k \; y[k-m]$$

Note that *M* and *N* may be different

Another Direct Form IIR Realization

- **When $N = M$,**

$$Y(z) = b_0 X(z) + \sum_{m=1}^{N} \left(b_m X(z) + a_m Y(z)\right) z^{-m} = b_0 X(z) + \sum_{m=1}^{N} W_m(z)\, z^{-m}$$

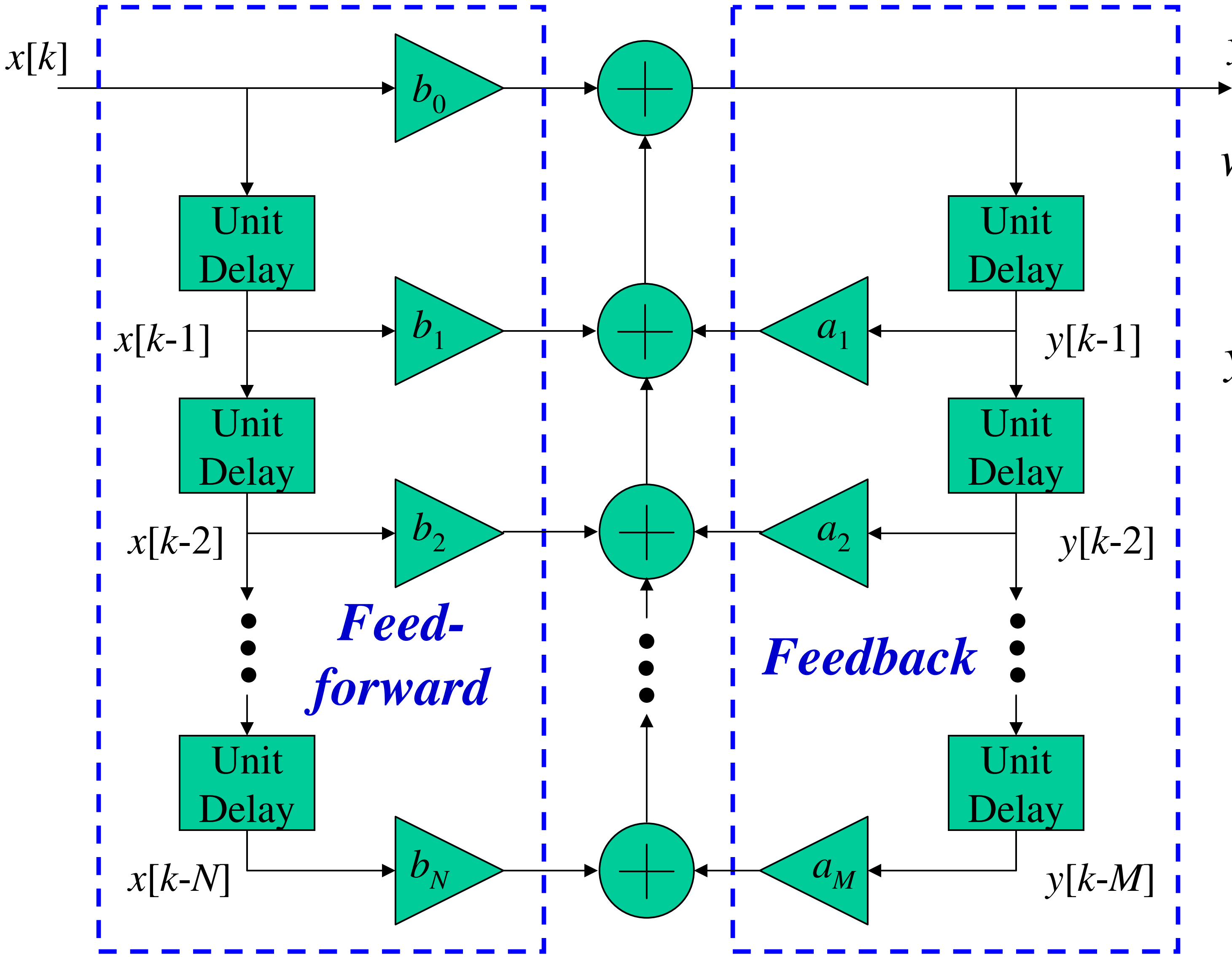
  – Here, $W_m(z) = b_m\, X(z) + a_m\, Y(z)$

  – In time domain, $w_m[k] = b_m x[k] + a_m y[k]$

$$y[k] = b_0 x[k] + \sum_{m=1}^{M} w_m[k-m]$$

- **Implementation complexity**
  – Computation: $M + N + 1$ MACs
  – Memory: $M + N$ words for previous inputs/outputs and $M + N + 1$ words for coefficients
- **More regular layout for hardware design**

Filter Structure As a Block Diagram



$$w_m[k] = b_m x[k] + a_m y[k]$$

$$y[k] = b_0 x[k] + \sum_{m=1}^{M} w_m[k-m]$$

Note that $M = N$ implied but can be different

Yet Another Direct Form IIR

- **Rearrange transfer function to be cascade of an an all-pole IIR filter followed by an FIR filter**

$$Y(z) = \frac{X(z)B(z)}{A(z)} = V(z)B(z) \text{ where } V(z) = \frac{X(z)}{A(z)}$$

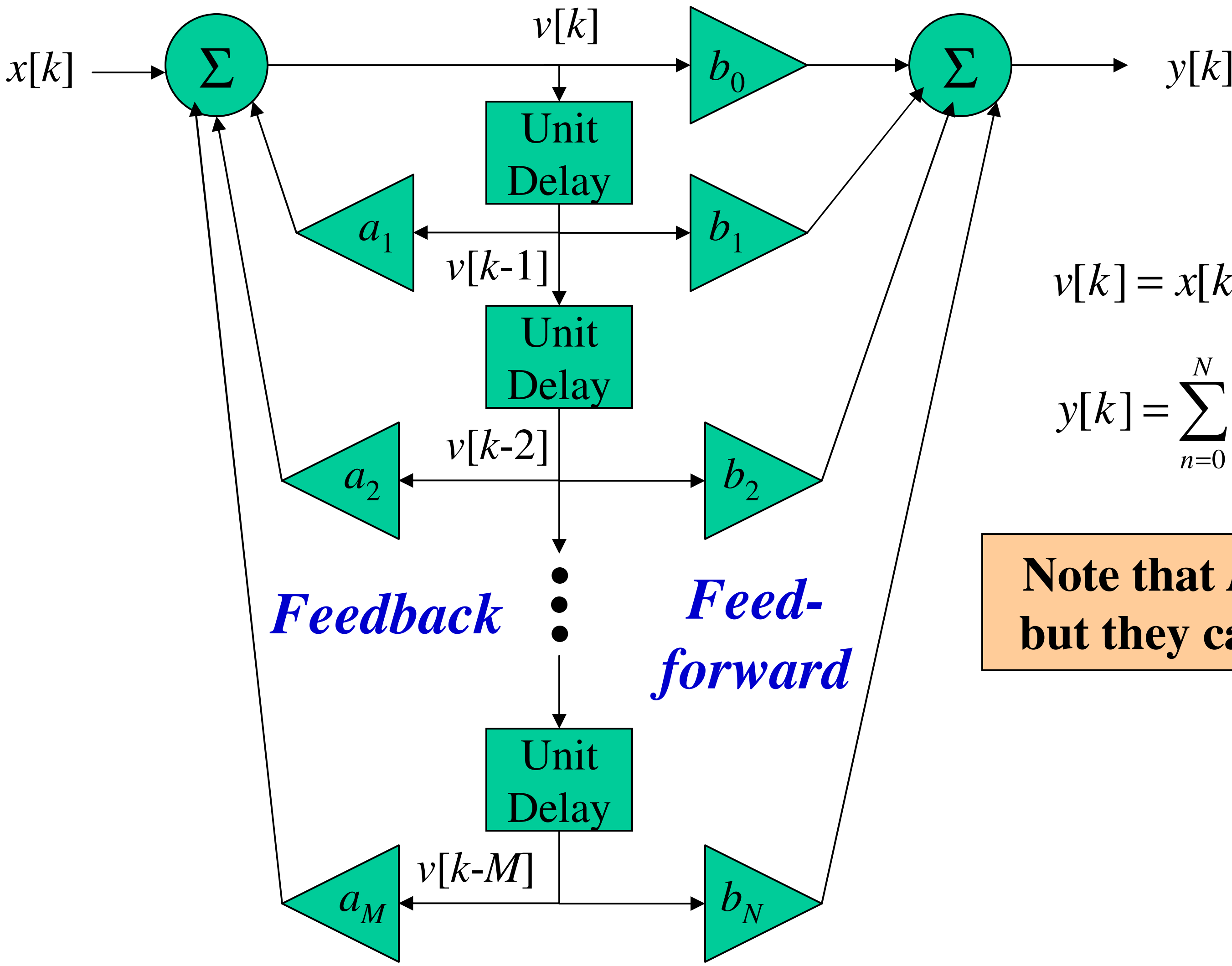
  – Here, $v[k]$ is the output of an all-pole filter applied to $x[k]$:

$$v[k] = x[k] + \sum_{m=1}^{M} a_m \, v[k-m]$$

$$y[k] = \sum_{n=0}^{N} b_n \, v[k-n]$$

- **Implementation complexity (assuming $M \geq N$)**
  – Computation: $M + N + 1 = 2N + 1$ MACs
  – Memory: $M + 1$ words for current/past values of $v[k]$ and $M + N + 1 = 2N + 1$ words for coefficients

Filter Structure As Block Diagram



$$v[k] = x[k] + \sum_{m=1}^{M} a_m \, v[k-m]$$

$$y[k] = \sum_{n=0}^{N} b_n \, v[k-n]$$

Note that $M = N$ implied but they can be different

M=2 yields a biquad

Demonstrations (*DSP First*)

- **Web site:** http://users.ece.gatech.edu/~dspfirst
- **Chapter 8: IIR Filtering Tutorial (Link)**
- **Chapter 8: Connection Betweeen the Z and Frequency Domains (Link)**
- **Chapter 8: Time/Frequency/Z Domain Moves for IIR Filters (Link)**

Stability

- **A digital filter is *bounded-input bounded-output (BIBO) stable* if for any bounded input $x[k]$ such that $|\, x[k]\, | \le B < \infty$, then the filter response $y[k]$ is also bounded $|\, y[k]\, | \le B < \infty$**
- **Proposition: A digital filter with an impulse response of $h[k]$ is BIBO stable if and only if**

$$\sum_{n=-\infty}^{\infty} |\, h[k]\, | < \infty$$

  - Any FIR filter is stable
  - A rational causal IIR filter is stable if and only if its poles lie inside the unit circle

Stability

- ***Rule #1*: For a causal sequence, poles are inside the unit circle (applies to *z*-transform functions that are ratios of two polynomials) OR**
- ***Rule #2*: Unit circle is in the region of convergence. (In continuous-time, imaginary axis would be in region of convergence of Laplace transform.)**
- **Example:** $a^k\, u[k] \overset{z}{\leftrightarrow} \dfrac{1}{1 - a\, z^{-1}}$ for $|z| > |a|$ pole at z=a

Stable if $|a| < 1$ by *rule #1* or equivalently

Stable if $|a| < 1$ by *rule #2* because $|z|>|a|$ and $|a|<1$

Z and Laplace Transforms

- **Transform difference/differential equations into algebraic equations that are easier to solve**
- **Are complex-valued functions of a complex frequency variable**

Laplace: $s = \sigma + j\,2\,\pi\,f$

Z: $z = r\,e^{j\,\omega}$

- **Transform kernels are complex exponentials: eigenfunctions of linear time-invariant systems**

Laplace: $e^{-s\,t} = e^{-\sigma\,t\,-\,j\,2\,\pi\,f\,t} = e^{-\sigma\,t}\;e^{-j\,2\,\pi\,f\,t}$

Z: $z^{-k} = (r\,e^{j\,\omega})^{-k} = r^{-k}\;e^{-j\,\omega\,k}$

dampening factor **oscillation term**

Z and Laplace Transforms

- **No unique mapping from Z to Laplace domain or from Laplace to Z domain**
  - Mapping one complex domain to another is not unique
- **One possible mapping is impulse invariance**
  - Make impulse response of a discrete-time linear time-invariant (LTI) system be a sampled version of the continuous-time LTI system.

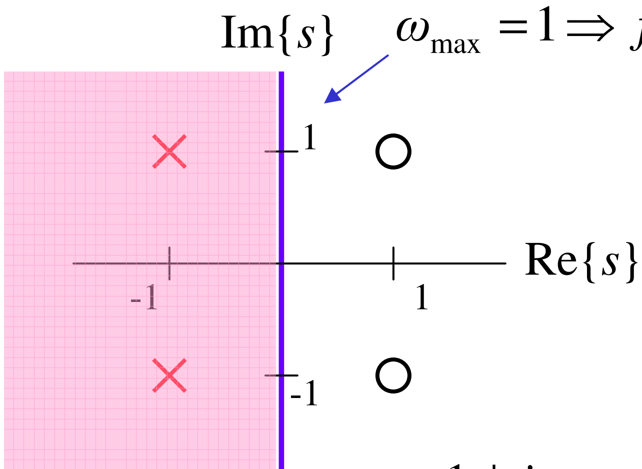
Z | **Laplace**

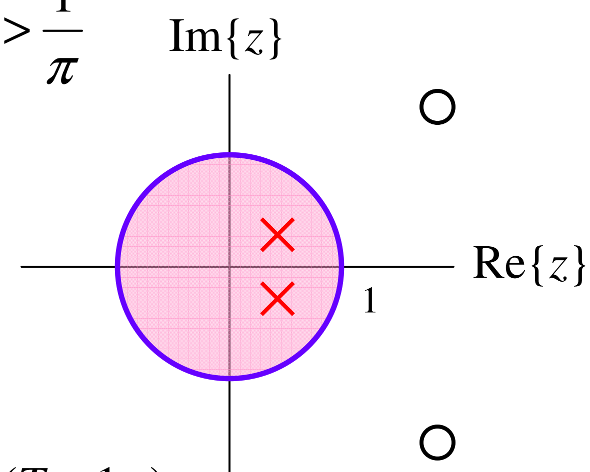
$$H(s) = H(z)\,|_{z=e^{sT}}$$

Impulse Invariance Mapping

- **Impulse invariance mapping is** $z = e^{sT}$

$$\omega_{max} = 1 \Rightarrow f_{max} = \frac{1}{2\pi} \Rightarrow f_s > \frac{1}{\pi}$$





$$s = -1 \pm j \Rightarrow z = 0.198 \pm j\,0.31 \;(T = 1 \text{ s})$$

$$s = 1 \pm j \Rightarrow z = 1.469 \pm j\,2.287 \;(T = 1 \text{ s})$$

$s = j\,2\,\pi\,f$

Laplace Domain	***Z Domain***
Left-hand plane	**Inside unit circle**
Imaginary axis	**Unit circle**
Right-hand plane	**Outside unit circle**

Impulse Invariance Derivation

$$\tilde{f}(t) = \sum_{k=0}^{\infty} f[k]\,\delta(t - kT)$$

$$\tilde{y}(t) = \sum_{k=0}^{\infty} y[k]\,\delta(t - kT)$$

$$\tilde{Y}(s) = H(s)\tilde{F}(s)$$

$$\sum_{k=0}^{\infty} y[k]\left(e^{sT}\right)^{-k} = H(s)\sum_{k=0}^{\infty} f[k]e^{-kTs}$$

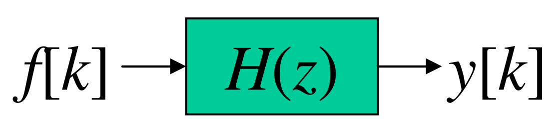
Let $z = e^{sT}$:

$$\sum_{k=o}^{\infty} y[k]z^{-k} = H(z)\sum_{k=0}^{\infty} f[k]z^{-k}$$

$$Y(z) = H(z)\,F(z)$$

Laplace

Z



Analog IIR Biquad

- **Second-order filter section with 2 poles and 2 zeros**
  - Transfer function is a ratio of two real-valued polynomials
  - Poles and zeros occur in conjugate symmetric pairs
- ***Quality factor*: technology independent measure of sensitivity of pole locations to perturbations**
  - For an analog biquad with poles at $a \pm j\, b$, where $a < 0$,

$$Q = \frac{\sqrt{a^2 + b^2}}{-2a} \quad \text{where} \; \frac{1}{2} \leq Q < \infty$$

  - Real poles: $b = 0$ so $Q = ½$ (exponential decay response)
  - Imaginary poles: $a = 0$ so $Q = \infty$ (oscillatory response)

Analog IIR Biquad

- **Impulse response with biquad with poles $a \pm j\, b$ with $a < 0$ but no zeroes:** $h(t) = C\, e^{a\,t} \cos(b\ t + \theta)$
  - Pure sinusoid when $a = 0$ and pure decay when $b = 0$
- **Breadboard implementation**
  - Consider a single pole at $-1/(R\ C)$. With 1% tolerance on breadboard R and C values, tolerance of pole location is 2%
  - *How many decimal digits correspond to 2% tolerance?*
  - *How many bits correspond to 2% tolerance?*
  - Maximum quality factor is about 25 for implementation of analog filters using breadboard resistors and capacitors.
  - Switched capacitor filters: $Q_{\max} \approx 40$ (tolerance $\approx 0.2\%$)
  - Integrated circuit implementations can achieve $Q_{\max} \approx 80$

Digital IIR Biquad

- **For poles at $a \pm j\, b = r\, e^{\pm j\,\theta}$, where $r = \sqrt{a^2 + b^2}$ is the pole radius (r < 1 for stability), with $y = -2\, a$:**

$$Q = \frac{\sqrt{(1+r^2)^2 - y^2}}{2\,(1-r^2)} \quad \text{where} \quad \frac{1}{2} \le Q < \infty$$

  - Real poles: $b = 0$ so $r = |\, a \,|$ and $y = \pm 2\, r$ which yields Q = ½ (exponential decay response $C_0\, a^n\, u[n] + C_1\, n\, a^n\, u[n]$)
  - Poles on unit circle: $r = 1$ so $Q = \infty$ (oscillatory response)
  - Imaginary poles: $a = 0$ so $y = 0$ so $$Q = \frac{1}{2}\,\frac{1+r^2}{1-r^2}$$
  - 16-bit fixed-point DSPs: $Q_{max} \approx 40$ (extended precision accumulators)

Analog/Digital IIR Implementation

- **Classical IIR filter designs**

  Filter of order n will have $n/2$ conjugate roots if n is even or one real root and $(n-1)/2$ conjugate roots if n is odd

  Response is very sensitive to perturbations in pole locations

- **Robust way to implement an IIR filter**

  Decompose IIR filter into second-order sections (biquads)

  Cascade biquads in order of ascending *quality factors*

  For each pair of conjugate symmetric poles in a biquad, conjugate zeroes should be chosen as those closest in Euclidean distance to the conjugate poles

Classical IIR Filter Design

- **Classical IIR filter designs differ in the shape of their magnitude responses**

  Butterworth: monotonically decreases in passband and stopband (no ripple)

  Chebyshev type I: monotonically decreases in passband but has ripples in the stopband

  Chebyshev type II: has ripples in passband but monotonically decreases in the stopband

  Elliptic: has ripples in passband and stopband

- **Classical IIR filters have poles and zeros, except that analog lowpass Butterworth filters are all-pole**
- **Classical filters have biquads with high Q factors**

Analog IIR Filter Optimization

- **Start with an existing (e.g. classical) filter design**
- **IIR filter optimization packages from UT Austin (in Matlab) simultaneously optimize**

  Magnitude response

  Linear phase in the passband

  Peak overshoot in the step response

  Quality factors

- **Web-based graphical user interface (developed as a senior design project) available at**

  http://signal.ece.utexas.edu/~bernitz

Analog IIR Filter Optimization

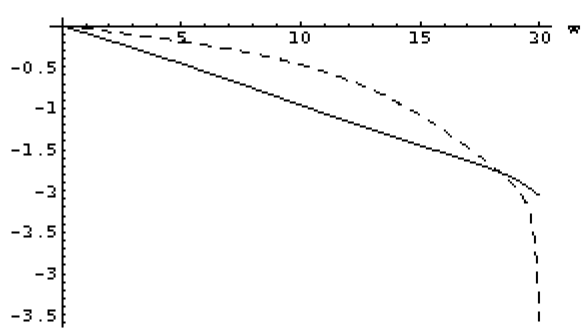
- **Design an analog lowpass IIR filter with $\delta_p = 0.21$ at $\omega_p = 20$ rad/s and $\delta_s = 0.31$ at $\omega_s = 30$ rad/s with**

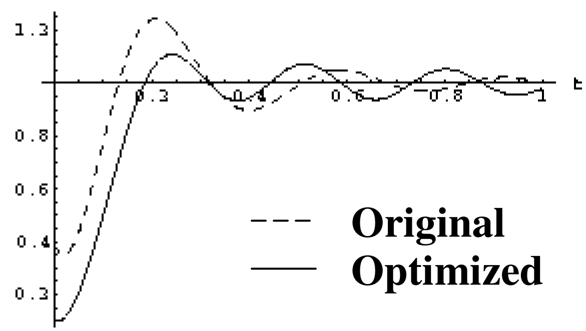
  Minimized deviation from linear phase in passband

  Minimized peak overshoot in step response

  Maximum quality factor of second-order sections is 10

Linearized phase in passband





Minimized peak overshoot

initial

Q	poles	zeros
1.7	$-5.3533 \pm j16.9547$	$0.0 \pm j20.2479$
61.0	$-0.1636 \pm j19.9899$	$0.0 \pm j28.0184$

optimized

Q	poles	zeros
0.68	$-11.4343 \pm j10.5092$	$-3.4232 \pm j28.6856$
10.00	$-1.0926 \pm j21.8241$	$-1.2725 \pm j35.5476$