Infinite Impulse Response Filters

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Digital IIR Filters

• Infinite Impulse Response (IIR) filter has impulse response of infinite duration, e.g.

\[ h[k] = \left( \frac{1}{2} \right)^k u[k] \quad \xrightarrow{Z} \quad H(z) = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k z^{-k} = \frac{1}{2} + \frac{1}{4} z^{-1} + \ldots = \frac{1}{1 - \frac{1}{2} z^{-1}} \]

• How to implement the IIR filter by computer?

Let \( x[k] \) be the input signal and \( y[k] \) the output signal,

\[ H(z) = \frac{Y(z)}{X(z)} \quad \Rightarrow \quad Y(z) = H(z)X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} X(z) \quad \Rightarrow \quad Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) \]

\[ Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) \quad \Rightarrow \quad y[k] - \frac{1}{2} y[k-1] = x[k] \quad \Rightarrow \quad y[k] = \frac{1}{2} y[k-1] + x[k] \]

Recursively compute output, given \( y[-1] \) and \( x[k] \)
Different Filter Representations

- **Difference equation**
  \[ y[k] = \frac{1}{2} y[k-1] + \frac{1}{8} y[k-2] + x[k] \]
  Recursive computation needs \(y[-1]\) and \(y[-2]\)
  For the filter to be LTI, \(y[-1] = 0\) and \(y[-2] = 0\)

- **Transfer function**
  Assumes LTI system
  \[ Y(z) = \frac{1}{2} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) + X(z) \]
  \[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1} - \frac{1}{8} z^{-2}} \]

- **Block diagram representation**
  Second-order filter section (a.k.a. biquad) with 2 poles and 0 zeros
  Poles at \(-0.183\) and \(+0.683\)
Digital IIR Biquad

- Two poles and zero, one, or two zeros

\[ v[k] \]
\[ \Sigma \]
\[ a_1 \]
\[ a_2 \]
\[ v[k] \]
\[ x[k] \]
\[ v[k-1] \]
\[ v[k-2] \]
\[ b_0 \]
\[ b_1 \]
\[ b_2 \]
\[ y[k] \]

- Take z-transform of biquad structure

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} \frac{Y(z)}{V(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}
\]

- Real coefficients \( a_1, a_2, b_0, b_1, \) and \( b_2 \) means poles and zeros in conjugate symmetric pairs \( \alpha \pm j \beta \)
Digital IIR Filter Design

- Poles near unit circle indicate filter’s passband(s)
- Zeros on/near unit circle indicate stopband(s)
- Biquad with zeros $z_0$ and $z_1$, and poles $p_0$ and $p_1$

Transfer function

$$H(z) = C \frac{(z-z_0)(z-z_1)}{(z-p_0)(z-p_1)}$$

Magnitude response

$$|H(e^{j\omega})| = \left| \frac{(e^{j\omega}-z_0)(e^{j\omega}-z_1)}{(e^{j\omega}-p_0)(e^{j\omega}-p_1)} \right|$$

$$|H(e^{j\omega})| = |C| \frac{|e^{j\omega}-z_0| |e^{j\omega}-z_1|}{|e^{j\omega}-p_0| |e^{j\omega}-p_1|}$$

$|a - b|$ is distance between complex numbers $a$ and $b$

Distance from point on unit circle $e^{j\omega}$ and pole location $p_0$
Digital IIR Biquad Design Examples

- Transfer function
  \[ H(z) = C \frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)} = C \frac{(1 - z_0 z^{-1})(1 - z_1 z^{-1})}{(1 - p_0 z^{-1})(1 - p_1 z^{-1})} \]

- Poles (X) & zeros (O) in conjugate symmetric pairs
  - For coefficients in unfactored transfer function to be real

- Filters below have what magnitude responses?

  ![Diagram of filter responses](image)
A Direct Form IIR Realization

- IIR filters having rational transfer functions

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}{1 - a_1 z^{-1} - \ldots - a_M z^{-M}} \quad \Rightarrow \quad Y(z) \left(1 - \sum_{m=1}^{M} a_m z^{-m}\right) = X(z) \sum_{n=0}^{N} b_n z^{-n}
\]

- Direct form realization
  - Dot product of vector of \(N + 1\) coefficients and vector of current input and previous \(N\) inputs (FIR section)
  - Dot product of vector of \(M\) coefficients and vector of previous \(M\) outputs (*FIR filtering of previous output values*)
  - Computation: \(M + N + 1\) MACs
  - Memory: \(M + N\) words for previous inputs/outputs and \(M + N + 1\) words for coefficients
Filter Structure As a Block Diagram

\[ y[k] = \sum_{n=0}^{N} b_k \ x[k-n] + \sum_{m=1}^{M} a_k \ y[k-m] \]

Note that \( M \) and \( N \) may be different
Another Direct Form IIR Realization

• When $N = M$,

$$Y(z) = b_0 X(z) + \sum_{m=1}^{N} \left( b_m X(z) + a_m Y(z) \right) z^{-m} = b_0 X(z) + \sum_{m=1}^{N} W_m(z) z^{-m}$$

  – Here, $W_m(z) = b_m \cdot X(z) + a_m \cdot Y(z)$
  – In time domain, $w_m[k] = b_m x[k] + a_m y[k]$

$$y[k] = b_0 x[k] + \sum_{m=1}^{M} w_m[k - m]$$

• Implementation complexity
  – Computation: $M + N + 1$ MACs
  – Memory: $M + N$ words for previous inputs/outputs and $M + N + 1$ words for coefficients

• More regular layout for hardware design
Filter Structure As a Block Diagram

\[ y[k] = b_0 x[k] + a_1 y[k-1] + \sum_{m=1}^{M} w_m[k-m] \]

\[ w_m[k] = b_m x[k] + a_m y[k] \]

Note that \( M = N \) implied but can be different.
Yet Another Direct Form IIR

• Rearrange transfer function to be cascade of an all-pole IIR filter followed by an FIR filter

\[ Y(z) = \frac{X(z)B(z)}{A(z)} = V(z)B(z) \text{ where } V(z) = \frac{X(z)}{A(z)} \]

  – Here, \( v[k] \) is the output of an all-pole filter applied to \( x[k] \):

\[ v[k] = x[k] + \sum_{m=1}^{M} a_m v[k-m] \]

\[ y[k] = \sum_{n=0}^{N} b_n v[k-n] \]

• Implementation complexity (assuming \( M \geq N \))

  – Computation: \( M + N + 1 = 2N + 1 \) MACs

  – Memory: \( M + 1 \) words for current/past values of \( v[k] \) and \( M + N + 1 = 2N + 1 \) words for coefficients
Filter Structure As Block Diagram

\[ x[k] \rightarrow \sum \rightarrow v[k] \rightarrow \sum \rightarrow y[k] \]

\[ v[k] = x[k] + \sum_{m=1}^{M} a_m v[k-m] \]

\[ y[k] = \sum_{n=0}^{N} b_n v[k-n] \]

\[ v[k] = [x[k], v[k-1], v[k-2], \ldots, v[k-M], y[k]] \]

\[ y[k] = [v[k], b_0 v[k], b_1 v[k], \ldots, b_N v[k]] \]

Note that \( M = N \) implied but they can be different

\( M=2 \) yields a biquad
Demonstrations (*DSP First*)

- **Web site:** [http://users.ece.gatech.edu/~dspfirst](http://users.ece.gatech.edu/~dspfirst)
- **Chapter 8: IIR Filtering Tutorial** ([Link](#))
- **Chapter 8: Connection Between the Z and Frequency Domains** ([Link](#))
- **Chapter 8: Time/Frequency/Z Domain Moves for IIR Filters** ([Link](#))
Stability

- A digital filter is *bounded-input bounded-output (BIBO) stable* if for any bounded input $x[k]$ such that $|x[k]| \leq B < \infty$, then the filter response $y[k]$ is also bounded $|y[k]| \leq B < \infty$

- Proposition: A digital filter with an impulse response of $h[k]$ is BIBO stable if and only if

\[ \sum_{n=-\infty}^{\infty} |h[k]| < \infty \]

- Any FIR filter is stable
- A rational causal IIR filter is stable if and only if its poles lie inside the unit circle
Stability

• **Rule #1**: For a causal sequence, poles are inside the unit circle (applies to $z$-transform functions that are ratios of two polynomials) OR

• **Rule #2**: Unit circle is in the region of convergence. (In continuous-time, imaginary axis would be in region of convergence of Laplace transform.)

• Example: $a^k u[k] \leftrightarrow \frac{1}{1 - a z^{-1}}$ for $|z| > |a|

Stable if $|a| < 1$ by rule #1 or equivalently

Stable if $|a| < 1$ by rule #2 because $|z| > |a|$ and $|a| < 1$
Z and Laplace Transforms

• Transform difference/differential equations into algebraic equations that are easier to solve

• Are complex-valued functions of a complex frequency variable

  Laplace:  \( s = \sigma + j \ 2 \pi f \)

  \( Z: \quad z = r \ e^{j \omega} \)

• Transform kernels are complex exponentials: eigenfunctions of linear time-invariant systems

  Laplace:  \( e^{-st} = e^{-\sigma t} - j2\pi f t = e^{-\sigma t} \quad e^{\omega k} \)

  \( Z: \quad z^{-k} = (r \ e^{j \omega})^{-k} = r^{-k} \quad e^{-j \omega k} \)

  dampening factor  oscillation term
Z and Laplace Transforms

• No unique mapping from Z to Laplace domain or from Laplace to Z domain
  – Mapping one complex domain to another is not unique

• One possible mapping is impulse invariance
  – Make impulse response of a discrete-time linear time-invariant (LTI) system be a sampled version of the continuous-time LTI system.

\[
\begin{align*}
Z & \quad \text{Laplace} \\
\quad f[k] & \quad \tilde{f}(t) \\
H(z) & \quad H(s) \\
y[k] & \quad \tilde{y}(t) \\
\end{align*}
\]

\[H(s) = H(z) \big|_{z = e^{sT}}\]
Impulse Invariance Mapping

- Impulse invariance mapping is \( z = e^{sT} \)

\[
\begin{align*}
\text{Im}\{s\} & \quad \omega_{\text{max}} = 1 \Rightarrow f_{\text{max}} = \frac{1}{2\pi} \Rightarrow f_s > \frac{1}{\pi} \\
\text{Re}\{s\} & \quad \text{Left-hand plane} \quad \text{Inside unit circle} \\
& \quad \text{Imaginary axis} \quad \text{Unit circle} \\
& \quad \text{Right-hand plane} \quad \text{Outside unit circle}
\end{align*}
\]

\[
\begin{align*}
s = -1 \pm j & \quad \Rightarrow z = 0.198 \pm j 0.31 \quad (T = 1 \text{ s}) \\
s = 1 \pm j & \quad \Rightarrow z = 1.469 \pm j 2.287 \quad (T = 1 \text{ s})
\end{align*}
\]
Optional

Impulse Invariance Derivation

\[ \tilde{f}(t) = \sum_{k=0}^{\infty} f[k] \delta(t - kT) \]
\[ \tilde{y}(t) = \sum_{k=0}^{\infty} y[k] \delta(t - kT) \]
\[ \tilde{Y}(s) = H(s) \tilde{F}(s) \]
\[ \sum_{k=0}^{\infty} y[k] (e^{sT})^{-k} = H(s) \sum_{k=0}^{\infty} f[k] e^{-kTs} \]

Let \( z = e^{sT} \):
\[ \sum_{k=0}^{\infty} y[k] z^{-k} = H(z) \sum_{k=0}^{\infty} f[k] z^{-k} \]
\[ Y(z) = H(z) F(z) \]
Analog IIR Biquad

• Second-order filter section with 2 poles and 2 zeros
  – Transfer function is a ratio of two real-valued polynomials
  – Poles and zeros occur in conjugate symmetric pairs

• Quality factor: technology independent measure of sensitivity of pole locations to perturbations
  – For an analog biquad with poles at \( a \pm j b \), where \( a < 0 \),
    \[
    Q = \frac{\sqrt{a^2 + b^2}}{-2a} \quad \text{where} \quad \frac{1}{2} \leq Q < \infty
    \]
  – Real poles: \( b = 0 \) so \( Q = \frac{1}{2} \) (exponential decay response)
  – Imaginary poles: \( a = 0 \) so \( Q = \infty \) (oscillatory response)
Analog IIR Biquad

• Impulse response with biquad with poles $a \pm j b$ with $a < 0$ but no zeroes: $h(t) = C e^{at} \cos(b \cdot t + \theta)$
  – Pure sinusoid when $a = 0$ and pure decay when $b = 0$

• Breadboard implementation
  – Consider a single pole at $-1/(R \cdot C)$. With 1% tolerance on breadboard $R$ and $C$ values, tolerance of pole location is 2%
  – *How many decimal digits correspond to 2% tolerance?*
  – *How many bits correspond to 2% tolerance?*
  – Maximum quality factor is about 25 for implementation of analog filters using breadboard resistors and capacitors.
  – Switched capacitor filters: $Q_{\text{max}} \approx 40$ (tolerance $\approx 0.2\%$)
  – Integrated circuit implementations can achieve $Q_{\text{max}} \approx 80$
Digital IIR Biquad

• For poles at $a \pm j b = r e^{\pm j \theta}$, where $r = \sqrt{a^2 + b^2}$ is the pole radius ($r < 1$ for stability), with $y = -2a$:

$$Q = \frac{\sqrt{(1+r^2)^2 - y^2}}{2(1-r^2)} \text{ where } \frac{1}{2} \leq Q < \infty$$

– Real poles: $b = 0$ so $r = |a|$ and $y = \pm 2r$ which yields $Q = \frac{1}{2}$ (exponential decay response $C_0 a^n u[n] + C_1 n a^n u[n]$)

– Poles on unit circle: $r = 1$ so $Q = \infty$ (oscillatory response)

– Imaginary poles: $a = 0$ so $y = 0$ so

$$Q = \frac{1}{2} \frac{1+r^2}{1-r^2}$$

– 16-bit fixed-point DSPs: $Q_{\text{max}} \approx 40$ (extended precision accumulators)
Analog/Digital IIR Implementation

• **Classical IIR filter designs**
  
  Filter of order $n$ will have $n/2$ conjugate roots if $n$ is even or one real root and $(n-1)/2$ conjugate roots if $n$ is odd
  
  Response is very sensitive to perturbations in pole locations

• **Robust way to implement an IIR filter**
  
  Decompose IIR filter into second-order sections (biquads)
  
  Cascade biquads in order of ascending *quality factors*
  
  For each pair of conjugate symmetric poles in a biquad, conjugate zeroes should be chosen as those closest in Euclidean distance to the conjugate poles
Classical IIR Filter Design

- Classical IIR filter designs differ in the shape of their magnitude responses
  - Butterworth: monotonically decreases in passband and stopband (no ripple)
  - Chebyshev type I: monotonically decreases in passband but has ripples in the stopband
  - Chebyshev type II: has ripples in passband but monotonically decreases in the stopband
  - Elliptic: has ripples in passband and stopband

- Classical IIR filters have poles and zeros, except that analog lowpass Butterworth filters are all-pole

- Classical filters have biquads with high Q factors
Analog IIR Filter Optimization

• Start with an existing (e.g. classical) filter design
• IIR filter optimization packages from UT Austin (in Matlab) simultaneously optimize
  - Magnitude response
  - Linear phase in the passband
  - Peak overshoot in the step response
  - Quality factors
• Web-based graphical user interface (developed as a senior design project) available at
  
  http://signal.ece.utexas.edu/~bernitz
Analog IIR Filter Optimization

- Design an analog lowpass IIR filter with $\delta_p = 0.21$ at $\omega_p = 20$ rad/s and $\delta_s = 0.31$ at $\omega_s = 30$ rad/s with
  - Minimized deviation from linear phase in passband
  - Minimized peak overshoot in step response
  - Maximum quality factor of second-order sections is 10

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