

# Fundamental Theorem of Linear Systems

**Theorem:** Let a linear time-invariant system  $g$  has an  $e_f(t)$  denote the complex sinusoid  $e^{j2\pi ft}$ . Then,  $g(e_f(\cdot), t) = g(e_f(\cdot), 0)e_f(t) = c e_f(t)$ .

**Example:** Analog RC Lowpass Filter

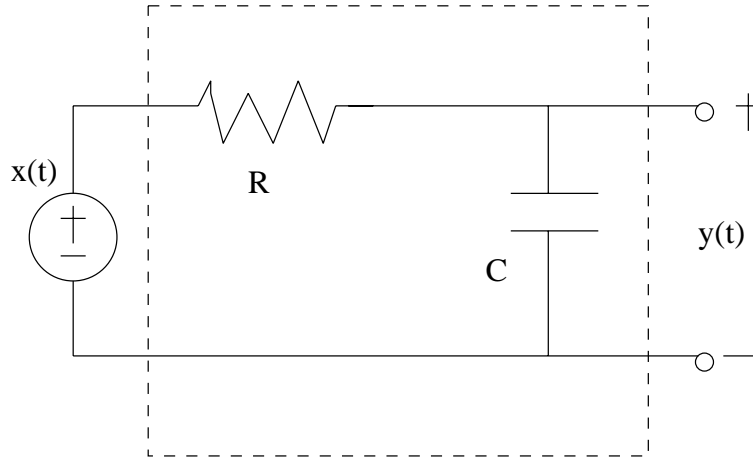


Figure 1: A First-Order Analog Lowpass Filter

The impulse response for the circuit in Fig. 1, i.e. the output measured at  $y(t)$  when  $x(t) = \delta(t)$ , is

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

For a complex sinusoidal input,  $x(t) = e_f(t) = e^{j2\pi ft}$ ,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t - \lambda) h(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} e^{j2\pi f(t-\lambda)} \frac{1}{RC} e^{-\frac{1}{RC}\lambda} u(\lambda) d\lambda \\ &= e^{j2\pi ft} \left[ \frac{1}{RC} \int_{-\infty}^{\infty} e^{-j2\pi f\lambda} e^{-\frac{1}{RC}\lambda} d\lambda \right] \\ &= \left[ \frac{\frac{1}{RC}}{j2\pi f + \frac{1}{RC}} \right] e^{j2\pi ft} \\ &= g(e_f(\cdot), 0) e_f(t) \end{aligned}$$

So,  $g(e_f(\cdot), 0) = H(f)$ , which is the transfer function of the system.