Wavelength Converters for a Packet Switch

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Large Devations for Classical Occupancy

- $\beta = r/n$ balls per urn thrown
- ξ fraction of urns to be occupied
- B_j^n number of balls needed to occupy the jth urn
- $R_n \equiv \sum_{j=1}^{b_n} B_j^n$ total balls needed

$$\begin{aligned} J(\xi, \beta) &\equiv \sup_t t\beta - \varphi(t) \\ &= (\beta - \xi)\log\rho + (1-\xi)\log(1-\xi) - \frac{(1-\rho\xi)}{\rho}\log(1-\rho\xi) \end{aligned}$$

where ρ satisfies

$$\beta\rho = -\log(1-\rho\xi) \quad .$$

Consider, $n = 100$ and $n - m = 40$ urns with $r = 90$ balls.

$$\begin{aligned} \beta &= 90/100 = 0.9 \\ \xi &= 0.4 \end{aligned}$$

Solving the fixed point

$$\begin{aligned} \rho &\approx 2.134 \\ Exact &\approx 1.48 \times 10^{-10} \\ Prob. &\approx 1.34 \times 10^{-9} \end{aligned}$$

Poisson approximation, $\lambda = ne^{-\beta} \approx 40.66$

$$Poisson = \sum_{j>m} e^{-\lambda}\frac{\lambda^j}{j!} \approx 0.0017$$

Switch Asymptotics I

Analysis of Loss due to Fibre Overflow

Packet Loss due to Overflow $= L^{(nF)}(a)$

$$\begin{aligned}
L^{(nF)}(a) &= \frac{1}{naF}\sum_{j=1}^{n} \mathbb{E}\left[[N_j - F]_+\right] & (1)\\
&= (aF)^{-1}\mathbb{E}\left[[N_1 - F]_+\right] \\
&= (aF)^{-1}\sum_{k=F+1}^{nF} \binom{nF}{k} (k-F)\left(\frac{a}{n}\right)^k (1-\frac{a}{n})^{nF-k} \\
&\approx (aF)^{-1}\sum_{k=F+1}^{\infty} \frac{(aF)^k}{k!} e^{-aF}(k-F) & (2)
\end{aligned}$$

if we assume equal traffic to each output fibre and then use the well known Poisson approximation to the binomial for large n.

Switch Asymptotics II

Probability of Insufficient Converters

Throwing $n\beta$ balls and occupying $n(\beta-\alpha)$ urns corresponds to $n\alpha$ converters being needed. Hence,

$$\mathbb{P}\{W \geq w | R = r\} \approx e^{-n(J(\beta-\alpha,\beta)}$$

Actually the number of packet arrivals (balls thrown) are random with (say) a binomial distribution. From the theorem of total probability we then have that,

$$\mathbb{P}\{W \geq w\} = \sum_r \mathbb{P}\{W \geq w | R = r\}\mathbb{P}\{R = r\}. \tag{3}$$

and since all the entries in the sum are exponentially small

$$\begin{aligned} \mathbb{P}\{W \geq w\} &\approx \max_r \mathbb{P}\{W \geq w | R = r\}\mathbb{P}\{R = r\} \\ &\approx e^{-n[\inf_{\alpha<\beta}(J(\beta-\alpha,\beta)+l(\beta,a))]} \end{aligned} \tag{4}$$

where the infimum in the exponent is over β with α fixed. $l(\beta, a)$ is the rate function for the binomial,

$$l(\beta, a) \equiv \beta \log \frac{\beta}{a} + (1-\beta) \log \frac{(1-\beta)}{(1-a)} .$$

Thus the large deviations exponent for random packet arrivals and a single wavelength is seen to be,

$$\mathcal{E}(\alpha) \equiv \inf_{\beta \geq \alpha} J(\beta-\alpha, \beta) + l(\beta, a). \tag{5}$$

Switch Asymptotics III

Overall Packet Loss

- $\beta = r/n$ packets per output per frequency
- α_w converters per output per frequency available

$$\begin{aligned}
(naF)^{-1}\mathbb{E}[N_w] &= (naF)^{-1}\sum_{k=w+1}^{nF}\mathbb{P}\{W \geq k\} \\
&= (anF)^{-1}\sum_{k=w+1}^{nF}\mathbb{P}\{\frac{W}{nF} \geq \frac{k}{nF}\} \\
&\approx (anF)^{-1}\sum_{k=w+1}^{nF} e^{-nF\mathcal{E}(\frac{k}{nF})} \\
&\approx a^{-1}\int_{\alpha=\alpha_w}^{1} e^{-nF\mathcal{E}(\alpha)}d\alpha \\
&\leq a^{-1}\int_{\alpha_w}^{\infty} e^{-nF(\mathcal{E}(\alpha_w)+(\alpha-\alpha_w)\mathcal{E}'(\alpha_w))}d\alpha \\
&= \frac{e^{-nF\mathcal{E}(\alpha_w)}}{naF\mathcal{E}'(\alpha_w)}
\end{aligned}$$

where we have used the fact that $\mathcal{E}(\alpha)$ is increasing and convex to lower bound the exponent by the tangent at α_w. We thus have

$$\mathbb{P}\{Packet\ Lost\} \leq L^{(nF)}(a) + \frac{e^{-nF\mathcal{E}(\alpha_w)}}{naF\mathcal{E}'(\alpha_w)}$$

Results I

Illustrating Exponent Trade-Off