(a) Explain what the system does.

The interesting part of the system, that is, from between the rate converters on the left to between the rate converters on the right, is effectively a TDM (time division multiplex) to FDM (frequency division multiplex) converter, and back again. It multiplexes several signals onto one channel by spacing them in frequency, and then demodulates them again without crosstalk.

(b) Transform relationships.

We have

$$\begin{aligned} W(z) &= H_0(z)U_0(z^2) + H_1(z)U_1(z^2) \\ \Rightarrow\ W(\sqrt{z}) &= H_0(\sqrt{z})U_0(z) + H_1(\sqrt{z})U_1(z)\ . \end{aligned} \tag{1}$$

Similarly, we can write

$$\begin{bmatrix} V_0(z) \\ V_1(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} G_0(\sqrt{z}) & G_0(-\sqrt{z}) \\ G_1(\sqrt{z}) & G_1(-\sqrt{z}) \end{bmatrix} \begin{bmatrix} W(\sqrt{z}) \\ W(-\sqrt{z}) \end{bmatrix}\ . \tag{2}$$

From (1) we have

$$\begin{bmatrix} W(\sqrt{z}) \\ W(-\sqrt{z}) \end{bmatrix} = \begin{bmatrix} H_0(\sqrt{z}) & H_1(\sqrt{z}) \\ H_0(-\sqrt{z}) & H_1(-\sqrt{z}) \end{bmatrix} \begin{bmatrix} U_0(z) \\ U_1(z) \end{bmatrix}\ . \tag{3}$$

Combining (2) and (3), we arrive at the relation

$$\begin{bmatrix} V_0(z) \\ V_1(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} G_0(\sqrt{z}) & G_0(-\sqrt{z}) \\ G_1(\sqrt{z}) & G_1(-\sqrt{z}) \end{bmatrix} \begin{bmatrix} H_0(\sqrt{z}) & H_1(\sqrt{z}) \\ H_0(-\sqrt{z}) & H_1(-\sqrt{z}) \end{bmatrix} \begin{bmatrix} U_0(z) \\ U_1(z) \end{bmatrix}\ . \tag{4}$$

The matrix form is enormously easier to deal with in this context.

(c) Eliminating crosstalk.

The off-diagonal elements of the GH matrix of (4) must be non-zero:

$$\begin{aligned} G_0(\sqrt{z})H_1(\sqrt{z}) + G_0(-\sqrt{z})H_1(-\sqrt{z}) &= 0 \\ G_1(\sqrt{z})H_0(\sqrt{z}) + G_1(-\sqrt{z})H_0(-\sqrt{z}) &= 0\ . \end{aligned} \tag{5}$$

This is satisfied if $H_0(z) = 2z^{-1}G_1(-z)$ and $H_1(z) = -2z^{-1}G_0(-z)$, as substitution into (5) will show.

(d) Perfect reconstruction.

The diagonal elements of the GH matrix of (4) must be unity:

$$\begin{aligned}\tfrac{1}{2}\left[G_0(\sqrt{z})H_0(\sqrt{z})+G_0(-\sqrt{z})H_0(-\sqrt{z})\right]=1\\ \tfrac{1}{2}\left[G_1(\sqrt{z})H_1(\sqrt{z})+G_1(-\sqrt{z})H_1(-\sqrt{z})\right]=1\ .\end{aligned} \qquad (6)$$

We choose $G_0(z)$ and $G_1(z)$ to be CQFs:

$$G_1(z)=-G_0(-z^{-1})z^{-N}\ . \qquad (7)$$

By substitution in (6), we arrive at the result

$$\left|G_0(\sqrt{z})\right|^2+\left|G_0(-\sqrt{z})\right|^2=1 \qquad (8)$$

for perfect reconstruction. Note that this is different from the requirement for PR in the notes for filter banks.