EE 381K **PROBLEM OF THE WEEK #3** **SPRING 1998**

Find the condition on the sampling period *T* such that $y_d(n) = z_d(n)$ for all *n*.

The analog output $y_a(t)$ of system 1 is given by $y(t) = h(t) * x(t)$, and therefore $Y(\Omega) = X(\Omega)H(\Omega)$. Thus the digital output $y_d(n)$ is given by

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left(\frac{w-2\pi m}{T}\right)H\left(\frac{w-2\pi m}{T}\right) \ . \tag{1}$$

Similarly, the digital output of system 2 is given by $z_d(n) = Th_d(n) * x_d(n)$, and therefore, using the same principles as (1), we get

$$Z(e^{j\omega}) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left(\frac{\omega-2\pi m}{T}\right)\sum_{n=-\infty}^{\infty} H\left(\frac{\omega-2\pi n}{T}\right) \ . \tag{2}$$

We must show the condition on T such that (1) and (2) are equal for all ω. When $m = n$, (1) and (2) are equal; the problem is therefore reduced to finding a T such that, for all ω,

$$H\left(\frac{\omega-2\pi m}{T}\right)X\left(\frac{\omega-2\pi n}{T}\right) = 0 \ . \tag{3}$$

Now we know that

$$\begin{aligned} X(\Omega) &= 0, \quad |\Omega| > \Omega_1 \\ H(\Omega) &= 0, \quad |\Omega| > \Omega_2 \end{aligned} \tag{4}$$

and therefore we see that

$$\begin{aligned} H\left(\frac{\omega-2\pi m}{T}\right) = 0 \quad &\Rightarrow \quad \left|\frac{\omega-2\pi m}{T}\right| > \Omega_2 \\ X\left(\frac{\omega-2\pi n}{T}\right) = 0 \quad &\Rightarrow \quad \left|\frac{\omega-2\pi n}{T}\right| > \Omega_1 \end{aligned} \tag{5}$$

for all m, n. Assuming the positive case (the negative case is identical), we see from (5) that

$$\omega - 2\pi m > \Omega_2 T, \quad \omega - 2\pi n < \Omega_1 T \ . \tag{6}$$

Rearranging and combining these two equations, we get

$$\begin{aligned} &\omega > \Omega_2 T + 2\pi m, \quad \omega < -\Omega_1 T - 2\pi n \\ &\Rightarrow \Omega_2 T + 2\pi m < -\Omega_1 T - 2\pi n \\ &\Rightarrow T(\Omega_1 + \Omega_2) < 2\pi(m-n) \ . \end{aligned} \tag{7}$$

The worst case occurs when $m - n = 1$; we then arrive at the result

$$T < \frac{2\pi}{\Omega_1 + \Omega_2} \ . \tag{8}$$