EE 381K

PROBLEM OF THE WEEK #2

SPRING 1998

(a) Find $x(n)$ when $X(z)=\ln(1-2z)$, ROC: $|z|<\frac{1}{2}$.

We use the property (from Oppenheim and Schafer, p. 180):

$$nx[n] \overset{\text{F.T.}}{\leftrightarrow} -z\frac{dX(z)}{d(z)} \quad \text{(same ROC)} \tag{1}$$

and apply to this problem:

$$\begin{aligned} X(z) &= \ln(1-2z) \\ \Rightarrow -z\frac{dX}{dz} &= \frac{2z}{1-2z} = \frac{-1}{1-\frac{1}{2}z^{-1}}\ , \quad |z|<\tfrac{1}{2}. \end{aligned} \tag{2}$$

We now apply (1) to find

$$\begin{aligned} nx[n] &= -\left(\tfrac{1}{2}\right)^n u[-n-1] \\ \therefore x[n] &= \frac{1}{n2^n}u[-n-1]\ \ . \end{aligned} \tag{3}$$

(b) Find $x(n)$ when $X(z)=\ln(1-z^{-1})$, ROC: $|z|>1$ (amended).

We follow the same procedure as part (a):

$$\begin{aligned} X(z) &= \ln(1-z^{-1}) \\ \Rightarrow -z\frac{dX}{dz} &= \frac{-z^{-1}}{1-z^{-1}}\ , \quad |z|>1. \end{aligned} \tag{4}$$

Again apply (1) to find

$$\begin{aligned} nx[n] &= -u[n-1] \\ \therefore x[n] &= -\frac{1}{n}u[n-1]\ \ . \end{aligned} \tag{5}$$

(c) Find the Fourier transform of $y(n)=(-1)^n u(n)$ directly.

We know we cannot compute the Fourier sum directly, since geometric series of the form $\sum_{n=0}^{\infty} ar^n$ converge only for $|r|<1$. For the Fourier sum, we have $r=e^{-j\omega} \Rightarrow |r|=1$. Instead, let us define

$$Y(e^{j\omega}) = \lim_{a\to 0}\sum_{n=0}^{\infty}(-1)^n e^{-j\omega n}e^{-an} \tag{6}$$

with a approaching zero from above so that the sum converges:

$$Y(e^{j\omega}) = \lim_{a \to 0} \frac{1}{1 + e^{-j\omega} e^{-a}} \quad . \tag{7}$$

Now we still cannot take the limit, since the RHS will blow up at $\omega = (2k+1)\pi$ for integer k. Let us therefore cross-multiply:

$$\lim_{a \to 0} \left(1 + e^{-j\omega} e^{-a}\right) Y(e^{j\omega}) = 1 \tag{8}$$

thus avoiding the singularity. We can now take limits and divide by the factor on the LHS. Since the resulting RHS will be a distribution (or, to put it another way, it blows up for certain values of ω), we must also add a term in $\delta(1 + e^{-j\omega})$, i.e. the factor by which we are dividing the RHS. Note that this is equivalent to casting (8) in the form $\Omega F(\Omega) = \Omega G(\Omega)$, but with less fuss:

$$Y(e^{j\omega}) = \frac{1}{1 + e^{-j\omega}} + C\delta(1 + e^{-j\omega}) \quad . \tag{9}$$

Although this form is perfectly acceptable, it is common to express the delta function as a direct function of ω rather than as an exponential of ω, so we rewrite it as

$$Y(e^{j\omega}) = \frac{1}{1 + e^{-j\omega}} + C \sum_{k=-\infty}^{\infty} \delta(\omega + (2k+1)\pi) \quad . \tag{10}$$

We now find C by inverse Fourier transform. We know that $u(0) = 1$. Furthermore, the Fourier kernel to compute $u(0)$ is $e^{j\omega 0} = 1$. Thus we have

$$y(0) = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \frac{d\omega}{1 + e^{-j\omega}} + C \int_{-\pi}^{\pi} \delta(\omega - \pi) d\omega \right] = 1 \quad . \tag{11}$$

(Note that we replaced the sum of delta functions with the single delta that falls within the range of integration.) The second integral is unity, by definition. The first can be integrated directly, or can be split into real and imaginary parts:

$$\begin{aligned} \frac{1}{1 + e^{-j\omega}} &= \frac{e^{j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}} \\ &= \frac{\cos(\omega/2) + j\sin(\omega/2)}{2\cos(\omega/2)} \quad . \end{aligned} \tag{12}$$

The imaginary term is odd and therefore drops out. The real term is just equal to $\frac{1}{2}$; it therefore integrates to $[\omega/2]_{-\pi}^{\pi} = \pi$. Thus we have

$$\begin{aligned} y(0) &= \tfrac{1}{2\pi}[\pi + C] = 1 \\ \therefore C &= \pi \quad . \end{aligned} \tag{13}$$

Thus we arrive at the result

$$Y(e^{j\omega}) = \frac{1}{1 + e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + (2k+1)\pi) \quad . \tag{14}$$